On Random Points in the Unit Disk

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Abstract

Let n be a positive integer and $\lambda > 0$ a real number. Let V_n be a set of n points in the unit disk selected uniformly and independently at random. Define $G(\lambda, n)$ to be the graph with vertex set V_n, in which two vertices are adjacent if and only if their Euclidean distance is at most λ. We call this graph a *unit disk random graph*. Let $\lambda = c\sqrt{\ln n/n}$ and let X be the number of isolated points in $G(\lambda, n)$. We prove that almost always $X \sim n^{1-c^2}$ when $0 \le c < 1$. It is known that if $\lambda = \sqrt{(\ln n + \phi(n))/n}$ where $\phi(n) \to \infty$, then $G(\lambda, n)$ is connected. By extending a method of Penrose, we show that under the same condition on λ, there exists a constant K such that the diameter of $G(\lambda, n)$ is bounded above by $K \cdot 2/\lambda$. Furthermore, with a new geometric construction, we show that when $\lambda = c\sqrt{\ln n/n}$ and $c > 2.26164\cdots$, the diameter of $G(\lambda, n)$ is bounded by $(4 + o(1))/\lambda$; and we modify this construction to yield a function $c(\delta) > 0$ such that the diameter is at most $2(1 + \delta + o(1))/\lambda$ when $c > c(\delta)$.

1 Introduction

Let n be a positive integer and $\lambda > 0$ a real number. Let V_n be a set of n points in the unit disk $\mathcal{D}$ selected uniformly and independently at random. Define $G(\lambda, n)$ to be the graph with

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vertex set V_n, in which two vertices are adjacent if and only if their Euclidean distance is at most λ. We call this graph a *unit disk random graph*, following the notation of Chen and the second author [4], whose work on package routing algorithms in mobile ad hoc wireless networks motivated this paper. $G(\lambda, n)$ is one example of a *random geometric graph* (RGG), in which nearby points randomly selected from some metric space are connected. Besides its graph-theoretical interest, the unit disk random graph is the structural basis of many wireless communication network protocols [4, 13, 14].

Early optimization problems on RGG's (e.g., [8, 10, 12]) include finding the minimal spanning tree or the nearest neighbor graph. Recent emphasis on connectivity and routing has followed the development of mobile ad hoc wireless networks. These networks are formed by a group of mobile nodes which communicate with each other over a wireless channel without any centralized control. A typical requirement is that each node in the network have a path to every other node in the network, that is, that the network be connected. With this in mind, Gupta and Kumar [7] considered the problem of determining the critical power (i.e., λ) guaranteeing connectedness asymptotically. Numerical simulations on connectivity and the size of the largest connected component can be found in engineering and physics literatures [2, 5]. A rigorous mathematical treatment was given by Penrose [11], who proved for the random geometric graph in the unit cube that the hitting times of t-connectivity and having minimum degree t are equal asymptotically. Penrose's theorem is analogous to a well-known result in Erdős-Renyi random graph theory: If edges of the complete graph on n vertices are added in an order chosen uniformly at random from all $\binom{n}{2}!$ possibilities, then with high probability for n large the resulting graph becomes t-connected at the instant it achieves a minimum degree of t [3]. We recommend the monograph of Penrose [9] as an authority on the mathematical theory and applications of random geometric graphs.

As soon as $G(\lambda, n)$ is connected, it is not clear that the graph diameter is small, that is, close to $2/\lambda$, which is the Euclidean diameter of the unit disk divided by the length of the longest possible edge. The main result of our work is a bound of K/λ (where K is an absolute constant) on the asymptotic value of the diameter of $G(\lambda, n)$ as soon as it is connected, and a tighter bound for larger λ. This is important information, for example, for the design and analysis of any routing algorithm in a mobile wireless ad hoc network with $G(\lambda, n)$ as an underlying model. Our technique for showing that long straight paths exist with high probability in $G(\lambda, n)$ can be adapted to other metric spaces, that is, to RGG's generally.

For the rest of the paper, we will use the following notation. The vertices of $G(\lambda, n)$ are $V_n = \{v_1, \ldots, v_n\}$. We use $\|\cdot\|_2$ for the Euclidean norm, so that $\mathcal{D} := \{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$. Denote by O the center of $\mathcal{D}$. If $x \in \mathbb{R}$ and $r > 0$, then $B(x, r) := \{y \in \mathbb{R} : \|x - y\|_2 \leq r\}$ is the disk of radius r about x. The distance between a point $x \in \mathbb{R}^2$ and a subset $S \subseteq \mathbb{R}^2$ is $d(x, S) := \inf_{y \in S} \|y - x\|_2$, and the distance between $S, S' \subseteq \mathbb{R}^2$ is $d(S, S') := \inf_{y \in S} d(y, S')$. For our purposes, the ball about a subset $S \subseteq \mathbb{R}^2$ of radius r is $B(S, r) := \{x \in \mathbb{R}^2 : d(x, S) \leq r\}$, which is the same as $\cup_{s \in S} B(s, r)$ when $|S|$ is finite. We say that $G(\lambda, n)$ has a property P *almost always (a.a.)* if

$$\lim_{n \to \infty} \Pr[G(\lambda, n) \text{ has the property } P] = 1.$$

The organization of this paper is as follows. In Section 2, we quote a result of Dette and Henze which characterizes the probability of having no isolated vertices when $\lambda \sim \sqrt{\ln n/n}$, and we prove that *a.a.*, $G(\lambda, n)$ has $n^{1-c^2}(1+o(1))$ isolated vertices when $\lambda = c\sqrt{\ln n/n}$ and $0 \le c < 1$. From a result of Gupta and Kumar or a modification of Penrose, *a.a.*, $G(\lambda, n)$ is connected when it has no isolated points; in particular, when $\lambda = \sqrt{(\ln n + \phi(n))/n}$ for any nonnegative $\phi(n) \to \infty$, *a.a.*, $G(\lambda, n)$ is connected. In Section 3, we show that there is an absolute constant K ($\approx 129.27\cdots$) such that under the same condition on λ, *a.a.*, $G(\lambda, n)$ has diameter $\le K \cdot 2/\lambda$. We then introduce a geometric construction to show that if $\lambda = c\sqrt{\ln n/n}$ and $c > 2.26164\cdots$, then *a.a.*, the diameter of $G(\lambda, n)$ is $\le 4(1+o(1))/\lambda$. In fact, there is a function $c(\delta) > 0$ such that if $c > c(\delta)$, then *a.a.*, the diameter of $G(\lambda, n)$ is $\le 2(1+\delta+o(1))/\lambda$.

2 Isolated vertices and connectivity

In [6, Thm. 2.5(c)], Dette and Henz compute the limiting distribution of the minimum λ for which $G(\lambda, n)$ has no isolated vertices. We quote this result as Theorem 1 before computing the expected number of isolated vertices when $\lambda = c\sqrt{\ln n/n}$ and $0 \le c < 1$, and finally discussing how a result of Gupta and Kumar, or a modification of a result of Penrose imply that $\lambda = \sqrt{\ln n/n}$ is also the threshold for connectedness of $G(\lambda, n)$. In order to give Theorem 1 in its original form, define $G'(\lambda, n)$ in the same way as $G(\lambda, n)$ except that the n points are selected from the disk of *unit area* and radius $1/\sqrt{\pi}$; furthermore, for fixed n define $D_{n,1} := \min\{\mu : G'(\mu, n) \text{ has minimum degree } 1\}$.

Theorem 1 (Dette, Henze).

$$\pi n D_{n,1}^2 - \log n \xrightarrow{\mathcal{D}} Z,$$

where Z is a random variable with Gumbel extreme value distribution $\Pr[Z \le t] = \exp(-\exp(-t))$, and the symbol $\xrightarrow{\mathcal{D}}$ means convergence in distribution.

Note that $G'(\mu, n)$ has no isolated points if and only if $D_{n,1} \le \mu$. Theorem 1 implies that $\lim_{n\to\infty} \Pr[\, D_{n,1} \le \sqrt{(\ln n + \alpha)/(\pi n)}\,] = \exp(-\exp(-\alpha))$. By simple re-scaling from the disk with unit area to the unit disk $\mathcal{D}$, we obtain that when $\lambda = \sqrt{(\ln n + \alpha)/n}$,

$$\lim_{n\to\infty} \Pr[\, G(\lambda, n) \text{ has no isolated points}\,] = \exp(-\exp(-\alpha)).$$

We now use a standard second moment argument (cf. [1]) to compute the number of isolated vertices when λ is below the threshold.

Proposition 2. Let $\lambda = c\sqrt{\ln n/n}$, and denote by X the number of isolated vertices in the unit disk random graph $G(\lambda, n)$. If $0 \le c < 1$, then *a.a.*, $X \sim n^{1-c^2}$.

Proof. For any vertex $v_i \in V_n$, let A_i be the event that v_i is isolated in $G(\lambda, n)$, and let X_i be the indicator of A_i; i.e. , $X_i = 1$ if A_i occurs and 0 otherwise. Then $X = \sum_{i=1}^{n} X_i$. We first compute the expected value $E[X]$.

The vertex v_i is isolated in $G(\lambda, n)$ if and only if there are no other vertices in $B(v_i, \lambda) \cap \mathcal{D}$. If $\|v_i\|_2 \le 1-\lambda$, then $B(v_i, \lambda) \subseteq \mathcal{D}$. If $\|v_i\|_2 > 1-\lambda$, the area of $B(v_i, \lambda) \cap \mathcal{D}$ is at least $\frac{1}{2}\pi\lambda^2(1+O(\lambda))$. Hence,

$$(1-\lambda^2)^{n-1} \le \Pr[A_i] \le (1-\lambda)^2(1-\lambda^2)^{n-1} + (2\lambda - \lambda^2)\left(1 - \frac{\lambda^2}{2}(1+O(\lambda))\right)^{n-1}. \tag{1}$$

Using $1-x = e^{-x}(1+o(1))$ as $x \to 0$, and noting that $0 \le c < 1$, we have $\Pr[A_i] = e^{-\lambda^2 n}(1+o(1)) = n^{-c^2}(1+o(1))$. By linearity of expectation, $E[X] = n\Pr[A_i] \sim n^{1-c^2}$. To show additionally that *a.a.*, $X \sim n^{1-c^2}$, it is sufficient to show that $\mathrm{Var}[X] = o(E(X)^2)$ (cf. [1, Cor. 4.3.3]). Note that

$$\mathrm{Var}[X] = \sum_i \mathrm{Var}[X_i] + \sum_{i \ne j} \mathrm{Cov}(X_i, X_j) \le E[X] + n^2 \mathrm{Cov}(X_1, X_2),$$

and so it suffices to show that $\mathrm{Cov}(X_1, X_2) = \Pr[A_1 \cap A_2] - \Pr[A_1]\Pr[A_2]$ is $o(n^{-2c^2})$. From (1), we have $\Pr[A_1]\Pr[A_2] = n^{-2c^2}(1+o(1))$. We compute $\Pr[A_1 \cap A_2]$ according to the distance between v_1 and v_2. There are three cases.

1. $\|v_1 - v_2\|_2 \le \lambda$. In this case $\Pr[A_1] = \Pr[A_2] = 0$, which implies $\Pr[A_1 \cap A_2 \mid \|v_i - v_j\|_2 \le \lambda] = 0$.

2. $\lambda < \|v_1 - v_2\|_2 \le 2\lambda$. If $\|v_1\|_2 \le 1 - 3\lambda$, both $B(v_1, \lambda)$ and $B(v_2, \lambda)$ are completely within $\mathcal{D}$, the overlapping of $B(v_1, \lambda)$ and $B(v_2, \lambda)$ has area less than $(2\pi/3 - \sqrt{3}/2)\lambda^2$. In this case the area of $B(v_1, \lambda) \cup B(v_2, \lambda)$ is at least $(4\pi/3 + \sqrt{3}/2)\lambda^2$. If $\|v_1\|_2 > 1 - 3\lambda$, the area of $(B(v_1, \lambda) \cup B(v_2, \lambda)) \cap \mathcal{D}$ is no less than the area of $B(v_1, \lambda) \cap \mathcal{D}$, which is $\ge \frac{1}{2}\pi\lambda^2(1+O(\lambda))$. Altogether, we have

$$\begin{aligned}&\Pr[A_i \cap A_j \mid \lambda < \|v_i - v_j\|_2 \le 2\lambda] \\ &\quad\le\ (1-3\lambda)^2\left(1 - \frac{4}{3}\lambda^2 - \frac{\sqrt{3}}{2\pi}\lambda^2\right)^{n-2} + (6\lambda - 9\lambda^2)\left(1 - \frac{\lambda^2}{2}(1+O(\lambda))\right)^{n-2} \ =\ o(n^{-c^2}).\end{aligned}$$

3. $\|v_1 - v_2\|_2 > 2\lambda$. If both v_1, v_2 are in $B(O, 1-\lambda)$, then $\Pr[A_1 \cap A_2 \mid \|v_1 - v_2\|_2 > 2\lambda] \ge (1-\lambda)^4(1-2\lambda^2)^{n-2}$. Conditioning on the events that one or both of v_1, v_2 are not in $B(O, 1-\lambda)$, we have

$$\begin{aligned}\Pr[A_1 \cap A_2 \mid \|v_1 - v_2\|_2 > 2\lambda] \ &\le\ (1-\lambda)^4(1-2\lambda^2)^{n-2} \\ &+\ 2(2\lambda - \lambda^2)(1-\lambda)^2\left(1 - \frac{3}{2}\lambda^2(1+O(\lambda))\right)^{n-2} + (2\lambda - \lambda^2)^2\big(1 - \lambda^2(1+O(\lambda))\big)^{n-2},\end{aligned}$$

which leads to $\Pr[A_1 \cap A_2 | \|v_1 - v_2\|_2 > 2\lambda] = (1+o(1))n^{-2c^2}$.

Combining the above three cases, by using $\Pr[\lambda < \|v_i - v_j\|_2 < 2\lambda] \le 3\lambda^2$ and $1 - 4\lambda^2 \le \Pr[\|v_i - v_j\|_2 \ge 2\lambda] \le 1$, we obtain $\Pr[A_1 \cap A_2] = (1+o(1))n^{-2c^2}$, which implies that $\mathrm{Cov}(X_1, X_2) = o(n^{-2c^2})$ as desired. $\square$

Gupta and Kumar proved [7, Theorem 3.2] that in the disk with unit area the threshold for having no isolated vertices is also the threshold for connectedness, in the following sense, (stated after rescaling to the unit disk).

Theorem 3 (Gupta, Kumar). Let $\lambda = \sqrt{(\ln n + \phi(n))/n}$. Then $G(\lambda, n)$ is almost always connected iff $\phi(n) \to \infty$.

It is also true by exhaustive checking of [11, Thm. 1.1] in the setting of the unit disk and Euclidean distance, that the hitting times for connectivity and for having minimum degree 1 are almost always the same. Either way we see that if $\phi(n) \to \infty$ and $\lambda = \sqrt{(\ln n + \phi(n))/n}$, then *a.a.*, $G(\lambda, n)$ is connected.

3 Connectivity and graph diameter

In this section we prove in Theorem 4 that as soon as $G(\lambda, n)$ is connected, the diameter of $G(\lambda, n)$ is at most K/λ, where $K > 0$ is an absolute constant. When $\lambda = c\sqrt{\ln n/n}$ and $c > 2.26164\cdots$, a separate argument using a geometric construction yields a bound of $(4+o(1))/\lambda$ in Theorem 7. We also show in Corollary 8 how further increasing c leads to a better upper bound on the diameter.

Theorem 4. Let $\phi(n) \to \infty$ be nonnegative. There exists an absolute constant $K > 0$ such that if $\lambda \geq \sqrt{(\ln n + \phi(n))/n}$, then *a.a.*, the unit disk random graph $G(\lambda, n)$ is connected with diameter $< K \cdot 2/\lambda$.

The proof is based on Proposition 4 below. For any two points v_i, v_j in the unit disk $\mathcal{D}$, let

$$T_{i,j}(k) = \big[\text{convex hull of } (B(v_i, k\lambda) \cup B(v_j, k\lambda))\big] \cap \mathcal{D},$$

where the convex hull of $B(v_i, k\lambda) \cup B(v_j, k\lambda)$ is the intersection of all convex sets containing $B(v_i, k\lambda) \cup B(v_j, k\lambda)$. Figure 1 illustrates this convex hull when v_i, v_j are away from the boundary of $\mathcal{D}$. Let $A_n(k)$ be the event that there exist two points $v_i, v_j \in V_n$ such that (i) at least one point is inside $B(O, 1-(k+1)\lambda)$, and (ii) there is no path of $G(\lambda, n)$ that lies in $T_{i,j}(k)$ and connects v_i and v_j.

Proposition 5. Let $\phi(n) \to \infty$ be nonnegative. There exists an absolute constant $K_0 > 0$ such that if $\lambda \geq \sqrt{(\ln n + \phi(n))/n}$, then

$$\lim_{n\to\infty} \Pr[A_n(K_0)] = 0.$$

Proof. Suppose $A_n(k)$ occurs. For simplicity, write $T_{i,j}(k)$ as $T_{i,j}$. Let $S(v_j)$ be the set of points in $V_n \cap T_{i,j}$ that are connected to v_j by paths in $T_{i,j}$. Let $R(v_j)$ be the points in $V_n \cap T_{i,j} \setminus S(v_j)$. $B(S(v_j), \frac{\lambda}{2})$ is a connected subset of $\mathbb{R}^2$, which is disjoint with $B(R(v_j), \frac{\lambda}{2})$. Let D_1 be the closure of the connected component of $T_{i,j} \backslash B(S(v_j), \frac{\lambda}{2})$ containing v_i, and let D_2 denote the closure of $T_{i,j} \backslash D_1$. Finally, let $L = D_1 \cap D_2$. By the proof of Penrose [11, p. 162], L is connected. Geometrically, L is the boundary of D_1 inside $T_{i,j}$, which separates points v_i and v_j.

L is on the boundary of $B(S(v_j), \frac{\lambda}{2})$, so for any point w on L, $d(w, S(v_j)) = \frac{\lambda}{2}$. On the other hand, for any point $u \in R(v_j)$, if there exists a point $w \in L$ such that $\|u - w\|_2 \leq \frac{\lambda}{2}$, then $d(u, S(v_j)) \leq \lambda$, a contradiction. Hence there are no points $w \in V_n \cap T_{i,j}$ such that $d(w, L) < \frac{\lambda}{2}$. This implies that the open strip which is the interior of $B(L, \frac{\lambda}{2}) \cap T_{i,j}$ contains no points of V_n.

Let $M = \partial T_{i,j} \backslash \partial \mathcal{D}$, where ∂X denotes the boundary of X. If $d(L, M) > \frac{\lambda}{2}$, then $S(v_j)$ is indeed a connected component of $G(\lambda, n)$, making $G(\lambda, n)$ disconnected. By Theorem 3, this happens with probability approaching 0. Hence we can assume that $d(L, M) \leq \frac{\lambda}{2}$. Note that the intersection of L and the line segment $v_i v_j$ is nonempty. Therefore the Euclidean diameter of L is at least $(k-1)\lambda$.

Case 1. Suppose both points v_i, v_j are in $B(O, 1 - (k+1)\lambda)$. Then $T_{i,j}$ is the convex hull of $B(v_i, k\lambda) \cup B(v_j, k\lambda)$, and $d(T_{i,j}, \partial \mathcal{D}) \geq \lambda$. In this case the (open) strip $B(L, \frac{\lambda}{2})$ is completely contained in $\mathcal{D}$.

Take $\epsilon = \lambda/(4\sqrt{2})$, and cover $\mathcal{D}$ by squares of side ϵ. Let X be the union of all squares that intersect with $\mathcal{D}$. Following Penrose, let L_n denote the set of centers for squares in X, and for $z \in L_n$, let B_z be the closed square centered at z.

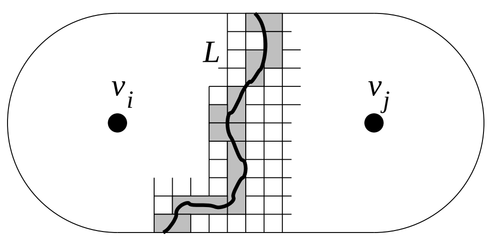

Figure 1: Both v_i, v_j are inside $B(O, 1 - (k+1)\lambda)$.

Let U^* be the set of points $z \in L_n$ such that B_z has nonempty intersection with L. Since L is connected, U^* is a $*$-connected subset of L_n, i.e., the union of the corresponding set of squares is connected (see Fig. 1). For each $z \in U^*$, the square B_z contains some point $u \in L$, and hence is contained in $B(u, \frac{\lambda}{2}) \subset B(L, \frac{\lambda}{2})$ This implies that for each $z \in U^*$, B_z is contained in $\mathcal{D}$ and has no points of V_n. By the Euclidean diameter condition on L, $\text{card}(U^*) \geq 4(k-1)$.

Case 2. Suppose the point v_j lies in $B(O, 1 - (k+1)\lambda)$, but v_i does not. Then part of the boundary of $T_{i,j}$ may be on the boundary of $\mathcal{D}$. In this case we need to consider the boundary effect.

Divide $T_{i,j}$ into two near-symmetric parts by the line $v_i v_j$. From the argument before Case 1, *a.a.*, there is a connected part of L, denoted by L', such that $d(L', M) \leq \frac{\lambda}{2}$, and L' only intersects the line segment $v_i v_j$ at one point w. Assume u is a point on M with $d(L', u) \leq \frac{\lambda}{2}$ (see Fig. 2).

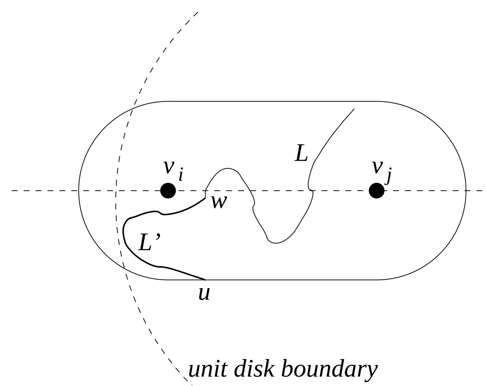

Figure 2: One of v_i, v_j is not inside $B(O, 1-(k+1)\lambda)$.

Consider the half of $T_{i,j}$ which contains L'. Since the boundary of $\mathcal{D}$ can only appear on one side of L', there is one half of the strip $B(L', \frac{\lambda}{2})$ that contains no points of $\partial\mathcal{D}$. Take L_1 to be the boundary of $B(L', \frac{\lambda}{4})$ in that half. Then the Euclidean diameter of L_1 is at least $(k-1)\lambda$, and $B(L', \frac{\lambda}{4})$ lies inside $\mathcal{D}$ and contains no points in V_n.

Again using the ϵ-square covering of $\mathcal{D}$, let W^* be the set of points $z \in L_n$ such that B_z has nonempty intersection with L_1. Then W^* is $*$-connected with cardinality at least $4(k-1)$. For each $z \in W^*$, B_z is contained in $\mathcal{D}$ and has no points in V_n.

Now for both cases, let $A_{n,i}$ denote the set of $*$-connected sets $A \subset L_n$ of cardinality i. By a Peierls argument [11, proof of Prop. 5.2], there exist constants $\gamma > 0$ and $c' > 0$ such that $\mathrm{card}(A_{n,i}) \leq c'\lambda^{-2}\exp(\gamma i)$. Hence

$$\begin{aligned}\Pr[A_n(k)] &\leq \sum_{i\geq 4(k-1)} \sum_{A\in A_{n,i}} \Pr[V_n \cap (\cup_{z\in A} B_z) = \emptyset] \\ &\leq \sum_{i\geq 4(k-1)} c'\lambda^{-2}\exp(\gamma i)\left(1-\frac{i}{\pi}\epsilon^2\right)^n \leq \sum_{i\geq 4(k-1)} c'\lambda^{-2}\exp\left(\gamma i - \frac{in\lambda^2}{32\pi}\right).\end{aligned}$$

For n sufficiently large, $\gamma < \frac{n\lambda^2}{64\pi}$, and $c'\lambda^{-2} < n$. Therefore

$$\Pr[A_n(k)] \leq n \sum_{i\geq 4(k-1)} \left[\exp\left(-\frac{n\lambda^2}{64\pi}\right)\right]^i = o\left(n^{1-\frac{k}{16\pi}}\right).$$

Taking $k = K_0 = 16\pi$, the last term in the above formula is $o(1)$. □

Proof of Theorem 4. Proposition 5 states that there exists an absolute constant $K_0 > 0$ such that as n goes to infinity, *a.a.* any two points v_i, v_j with at least one in $B(O, 1-(K_0+1)\lambda)$ are connected by a path inside $T_{i,j}(K_0)$. For points v_i, v_j with at least one in $B(O, 1-(K_0+1)\lambda)$, let

$$v_i = u_0 \longrightarrow u_1 \longrightarrow u_2 \longrightarrow \cdots \longrightarrow u_g = v_j$$

be such a path with minimum g. Then clearly the Euclidean distance between u_k and u_l is larger than λ for any $|k-l|>1$. This implies that the balls $\{B(u_k, \frac{\lambda}{2}) : k \text{ even}\}$ are disjoint. Hence

$$\begin{aligned}\left\lceil \frac{g}{2} \right\rceil \pi \left(\frac{\lambda}{2}\right)^2 &\le \text{Area}\left(B\left(T_{i,j}(K_0), \frac{\lambda}{2}\right)\right) \\ &\le \pi\left(K_0+\frac{1}{2}\right)^2 \lambda^2 + 2\left(K_0+\frac{1}{2}\right)\lambda, \end{aligned} \tag{2}$$

which gives path length

$$g \le \frac{8(2K_0+1)}{\pi\lambda} + K_1,$$

where K_1 is an absolute constant.

If both vertices lie outside of $B(O, 1-(K_0+1)\lambda)$, then we can travel from the first vertex to an intermediate vertex just inside $B(O, 1-(K_0+1)\lambda)$ using a constant length path, and then on to the second vertex. To this end, let $B_n(k)$ be the event that some vertex $v_i \notin B(O, 1-(k+1)\lambda)$ has no corresponding $v_j \in B(O, 1-(k+1)\lambda) \cap B(v_i, (2k+1)\lambda)$ with a path connecting v_i and v_j inside $T_{i,j}(k)$. We break this event into two cases as follows. Let $C_n(k)$ be the event that there exists a vertex $v_i \notin B(O, 1-(k+1)\lambda)$ with no corresponding vertex $v_j \in B(O, 1-(k+1)\lambda) \cap B(v_i, (2k+1)\lambda)$. Let $D_n(k)$ be the event that there exists a vertex $v_i \notin B(O, 1-(k+1)\lambda)$ with a corresponding vertex $v_j \in B(O, 1-(k+1)\lambda) \cap B(v_i, (2k+1)\lambda)$ but no connecting path in $T_{i,j}(k)$. We have

$$\begin{aligned}\Pr[C_n(k)] &\le n\left(1-(1-(k+1)\lambda)^2\right)\left(1-\left(\frac{k\lambda}{2}\right)^2\right)^{n-1} = o(\lambda n^{1-\frac{k^2}{4}}), \\ \Pr[D_n(k)] &\le \Pr[A_n(k)] \le o(n^{1-\frac{k}{16\pi}}).\end{aligned}$$

Therefore with $k = K_0 = 16\pi$ as in Proposition 5, we have

$$\Pr[B_n(K_0)] \le \Pr[C_n(K_0)] + \Pr[D_n(K_0)] = o(1).$$

This proves that *a.a.*, any point $v_i \notin B(O, 1-(K_0+1)\lambda)$ is connected to a point $v_j \in B(v_i, (2K_0+1)\lambda) \cap B(O, 1-(K_0+1)\lambda)$ by a path in $T_{i,j}(K_0)$. In this case, the area of $T_{i,j}(K_0)$ is no more than $\pi(K_0\lambda)^2 + 2K_0(2K_0+1)\lambda^2$. Thus there exists an absolute constant K_2 such that *a.a*, any vertex $v_i \notin B(O, 1-(K_0+1)\lambda)$ is connected to a vertex $v_j \in B(v_i, (2K_0+1)\lambda) \cap B(O, 1-(K_0+1)\lambda)$ by a path of length $\le K_2$. Combining with the previous argument, the diameter of $G(\lambda, n)$ is at most

$$\frac{4(2K_0+1)}{\pi} \cdot \frac{2}{\lambda} + K_3,$$

where $K_3 = K_2 + K_1$ is an absolute constant. □

In statement of Theorem 4, taking any $K > 128 + 4/\pi$ suffices, but we did not attempt to optimize K. We shall need later the following corollary, essentially obtained from the proof of Theorem 4 by replacing the Euclidean diameter 2 of the unit disk in (2) with $\|v_i - v_j\|_2$ for arbitrary vertices $v_i, v_j \in V_n$.

Corollary 6. Let $\phi(n) \to \infty$ be nonnegative. There exists an absolute constant $K > 0$ (the same as in Theorem 4) such that if $\lambda \geq \sqrt{(\ln n + \phi(n))/n}$, then *a.a.*, every pair of vertices v_i and v_j in the unit disk random graph $G(\lambda, n)$ is connected by a path of length $< K \cdot \|v_i - v_j\|_2/\lambda$.

Proof. The case $\|v_i - v_j\|_2 \leq \lambda$ is trivial. For $\|v_i - v_j\|_2 > \lambda$, let K_0 be as in Proposition 5 and let K_2 and K_1 be as in the proof of Theorem 4. If at least one of v_i, v_j is in $B(O, 1-(K_0+1), \lambda)$, the result follows immediately by replacing 2 with $\|v_i - v_j\|_2$ in (2). If both $v_i, v_j \notin B(O, 1-(K_0+1), \lambda)$, then there exists an intermediate vertex $v_k \in B(O, 1-(K_0+1)\lambda) \cap B(v_i, (2K_0+1)\lambda)$, a path of length at most K_2 between v_i and v_k, and a path of length at most $(4(2K_0+1)/\pi) \cdot \|v_k - v_j\|_2/\lambda + K_1$ between v_k and v_j. Since $\|v_k - v_j\|_2 \leq \|v_i - v_j\|_2 + \|v_i - v_k\|_2 \leq \|v_i - v_j\|_2 + (2K_0+1)\lambda$, the result follows, as $\lambda < \|v_i - v_j\|_2$. □

We now consider an overlay of the unit disk with columns of tiles of height $\lambda/2$ and width $\sqrt{3}\lambda/2$, where each column is centered on O and there is a uniform angular spacing between columns. Each tile has an "active" lens-shaped interior region, so that two vertices in the active regions of abutting tiles in the same column are adjacent in $G(\lambda, n)$. When $c > 2.26164\cdots$, *a.a.*, there is a vertex in V_n lying inside every tile's active region, guaranteeing paths in $G(\lambda, n)$ in each column with average edge length $\sim \lambda/2$, which forms the basis of the bound on the diameter of $G(\lambda, n)$ in the following theorem.

Theorem 7. Let $\lambda = c\sqrt{\ln n/n}$. If $c > \dfrac{\sqrt{12\pi}}{\sqrt{4\pi - 3\sqrt{3}}} \approx 2.26164\cdots$, then *a.a.*, the unit disk random graph $G(\lambda, n)$ is connected with diameter $\leq (4 + o(1))/\lambda$.

Proof. Construct an overlay of $\mathcal{D}$ with columns of identical rectangular tiles with height $\lambda/2$ and width $\sqrt{3}\lambda/2$. A single column of tiles, as illustrated in Figure 3(a), consists of as many tiles as will completely fit inside $\mathcal{D}$, arranged so that the geometric center of the middle tile coincides with O. The overlay of $\mathcal{D}$ consists of $t = 2\lfloor(\ln n)/2\rfloor$ such columns at a uniform angular spacing of π/t, as shown in Figure 3(b) (other choices of t are possible). Each tile has an "active" lens-shaped interior region. For the tile in Figure 3(a) bounded by lines L_1, L_2, L_3, L_4, this is the region $ABCD$, bounded by the arc ABC of the circle which has center D and radius $\lambda/2$, and the arc ADC of the circle with center B and radius $\lambda/2$. It is easy to see that the area of the region $ABCD$ is

$$\alpha = \frac{4\pi - 3\sqrt{3}}{24} \cdot \lambda^2.$$

By construction, two vertices in V_n are adjacent in $G(\lambda, n)$ (i) if they lie in the active regions of abutting tiles in the same column, or (ii) if they lie anywhere inside the same tile.

Let M denote the total number of tiles in all t columns. Noting that the height of a single tile is $\lambda/2$, we can compute that

$$M = t\left(\frac{2}{\lambda/2} + \Theta(1)\right) = \frac{4}{\lambda}\left(\ln n + \Theta(1)\right).$$

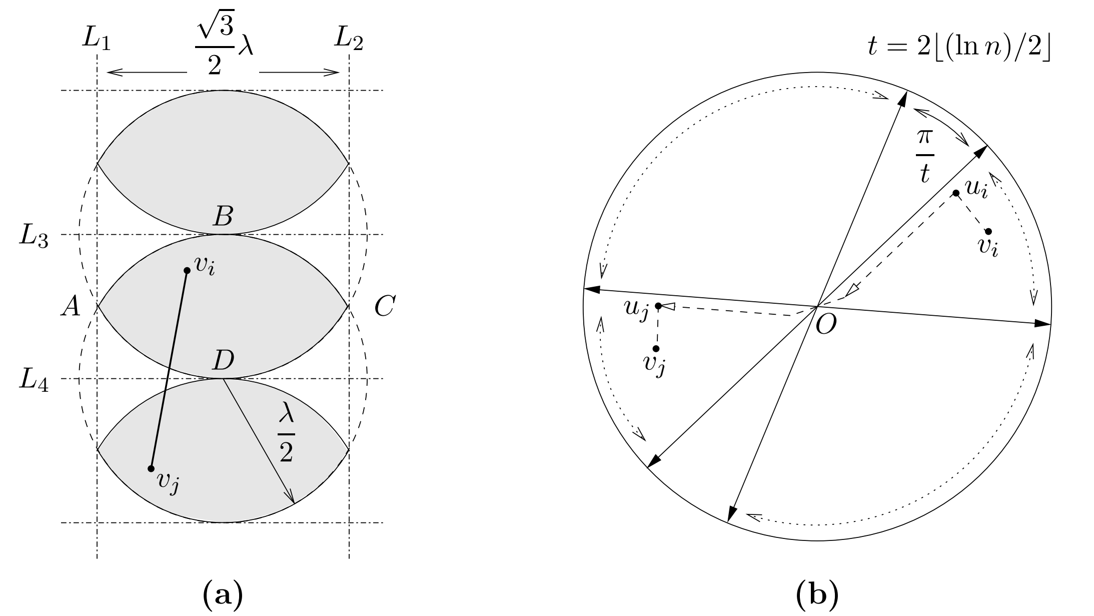

Figure 3: An overlay of the unit disk of **(a)** columns of tiles, such that **(b)** the $\sim \ln n$ columns are uniformly angularly spaced and each centered at O.

Let s be a tile counted in M. Let $F(s)$ denote the event that v_i is not in the active region s for all $i = 1, \dots, n$. Then, for each s,

$$\Pr[F(s)] \;=\; \left(1 - \frac{\alpha}{\pi}\right)^n \;\sim\; n^{-c^2 \cdot \frac{4\pi - 3\sqrt{3}}{24\pi}}.$$

Therefore choosing an arbitrary but fixed tile s_0 counted by M,

$$\begin{aligned}\Pr\left[\bigcup_{s \text{ counted in } M} F(s)\right] \;&\leq\; M \cdot \Pr[F(s_0)] \;\leq\; (1 + o(1))\frac{4 \ln n}{\lambda} n^{-c^2 \cdot \frac{4\pi - 3\sqrt{3}}{24\pi}} \\ &=\; \left(\frac{4}{c} + o(1)\right) \sqrt{\ln n}\, n^{\frac{1}{2} - c^2 \cdot \frac{4\pi - 3\sqrt{3}}{24\pi}}, \end{aligned} \tag{3}$$

which implies that, as $n \to \infty$,

$$\Pr\left[\bigcup_{s \text{ counted in } M} F(s)\right] \;=\; o(1)$$

provided that

$$c \;>\; \sqrt{\frac{24\pi}{2(4\pi - 3\sqrt{3})}} \;\approx\; 2.26164\cdots.$$

This means that *a.a.*, every tile counted in M contains at least one vertex of V in its active region.

Let τ_i and τ_j be the tilings with angular orientation closest to those of the lines through v_i and O, and v_j and O, respectively. Almost always, for any pair $v_i, v_j \in V_n$ there must exist vertices u_i, u_j in the active regions of tiles in τ_i and τ_j, respectively, such that $\|v_i - u_i\|_2$ and $\|v_j - u_j\|_2$ are at most $\pi/(2t) + 2\lambda$. By Corollary 6, there exist paths in $G(\lambda, n)$ from v_i to u_i, and from v_j to u_j, each of length at most $K(\pi/(2t) + 2\lambda)/\lambda = o(1)/\lambda$, where K is an absolute constant independent of the choice of v_i, v_j. Traveling from u_i to a tile in τ_i containing O takes at most $1/(\lambda/2)$ steps. Transferring to the (active region of the) tile in τ_j containing O takes at most 1 step, and traveling to u_j takes at most $1/(\lambda/2)$ steps. Therefore *a.a.*, the diameter of $G(\lambda, n)$ is at most $(4 + o(1))/\lambda$. □

For any two points with distance d in $\mathcal{D}$, the shortest path connecting them has length at least d/λ, which implies that asymptotically the diameter of the graph $G(\lambda, n)$ is $\geq (2 - o(1))/\lambda$. Combining this with Theorem 4, when $G(\lambda, n)$ is connected, its diameter is $\Theta(\lambda^{-1})$. We can improve Theorem 7 and approach the asymptotic lower bound by giving ground in terms of c in order to stretch the height of the tiles, as described in the following corollary.

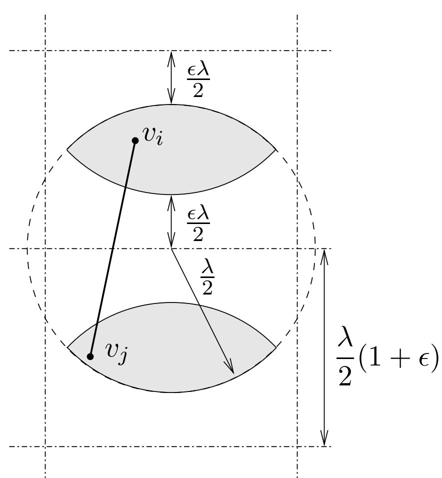

Figure 4: A stretched tile increases the length of the typical edge $\{v_i, v_j\}$ inspected when bounding the diameter of $G(\lambda, n)$.

Corollary 8. Let $\lambda = c\sqrt{\ln n/n}$. For every $\delta \in (0, 1]$, there exists a $c(\delta) > 0$ such that if $c > c(\delta)$, then *a.a.*, the unit disk random graph $G(\lambda, n)$ is connected with diameter $\leq 2(1 + \delta + o(1))/\lambda$.

Proof. Let $\delta \in (0, 1]$, and let $\epsilon = (1 - \delta)/(1 + \delta)$. Starting with the tiling of Theorem 7, increase the height of each tile by $\epsilon\lambda/2$ to obtain the new tiles illustrated in Figure 4. This is done while keeping the center tile in each row centered on O, and so that the tiling overlay consists only of those tiles entirely within $\mathcal{D}$. The width of a tile may still be taken to be $\sqrt{3}\lambda/2$, although a slightly greater width depending on ϵ is possible while still forcing adjacency between any two vertices in the same tile. Note that the two circles whose intersection determines one of the active regions are still centered on the boundaries of consecutive tiles, but the extra height of the tiles causes a vertical gap of $\epsilon\lambda/2$ between the active region and the tile boundary. The area of the active region as a function of ϵ is $\text{Area}(\epsilon) = \frac{\lambda^2}{2}\arccos\left(\frac{1}{2}(1+\epsilon)\right) - \frac{\lambda^2}{8}(1+\epsilon)\sqrt{3-2\epsilon-\epsilon^2}$. Now let $c > c(\delta) = \sqrt{\pi/(2\text{Area}(\epsilon)/\lambda^2)}$, making the quantity in (3) at most $o(1)$. Therefore *a.a.*, there is a point of V_n in the active region of each tile. Following the procedure in the proof of Theorem 7 to find a short path between any pair $v_i, v_j \in V_n$, *a.a.*, the diameter of $G(\lambda, n)$ is at most $\frac{2}{(\lambda/2)(1+\epsilon)} + o(1)/\lambda = 2(1+\delta+o(1))/\lambda$. □

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