Instructions: Do any 7 of the 9 problems in this exam. Show all of your work clearly. Please indicate which 2 of the 9 problems you are skipping.

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. Consider the function \( I(x) := \int_0^1 t^x(2-t)^x dt \). Use Laplace’s method and Watson’s lemma to find the asymptotic expansion \( I(x) \) to order \( O(x^{-2}) \), for \( x \to \infty \).

2. Let \( S \) be Schwartz space, \( S’ \) be the space of tempered distributions, and let \( T(x) = (x)_+ - 3(x - 2)_+ + 2(x - 3)_+ \), where \( (x)_+ = \begin{cases} x & 0 \leq x \\ 0 & x < 0 \end{cases} \).

   (a) Define \( S \) and give the semi-norm topology for it. In addition, define \( S’ \).

   (b) Find the distributional second derivative, \( T'' \), and the Fourier transform of \( T'' \). Use these results to find the Fourier transform of \( T \).

3. Consider a functional \( J[u] \), where \( u \in V \), and \( V \) is a Banach space.

   (a) Define the Frechét derivative and the Gâteaux derivative for \( J[u] \). Illustrate the difference between them with a simple two variable example.

   (b) Consider the constrained functional,

   \[
   J[u] = \int_0^1 pu^2 dx + \sigma u(1)^2, \quad H[u] = \int_0^1 u^2 dx = 1,
   \]

   where \( u \in C^{(1)}[0,1] \), \( u(0) = 0 \), and \( \sigma > 0 \). Calculate the variational derivative of the problem, using Lagrange multipliers. Find the Sturm-Liouville eigenvalue problem associated with it.
(c) How does the second eigenvalue of this problem compare with the second eigenvalue of the corresponding Dirichlet problem, i.e., \( u(0) = u(1) = 0 \)? Explain your answer.

4. Let \( \mathcal{H} \) be a Hilbert space with inner product and norm given by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \). You may assume that \( \mathcal{H} \) is a real Hilbert space.

(a) State and prove the Riesz Representation Theorem.

(b) Suppose that \( \mathcal{H} \subset C[0, 1] \) and that, for \( f \in \mathcal{H} \), \( \| f \|_{C[0, 1]} \leq \| f \| \).

Show that for every \( \xi \in [0, 1] \) there is a function \( K_\xi(\cdot) \in \mathcal{H} \) for which \( f(\xi) = \langle f, K_\xi \rangle \).

(c) Consider \( N \) distinct points \( 0 \leq \xi_1 < \xi_2 < \cdots < \xi_N \leq 1 \) and let \( \mathcal{U} := \text{span}\{K_{\xi_j}\}_{j=1}^N \), which is a finite dimensional subspace of \( \mathcal{H} \).

Show that for any \( f \in \mathcal{H} \), the orthogonal projection of \( f \) onto \( \mathcal{U} \), \( \tilde{f} = \text{Proj}_\mathcal{U} f \), satisfies \( \tilde{f}(\xi_j) = f(\xi_j), j = 1, \ldots, N \).

5. The degree \( n \) Chebyshev polynomial can be defined via the Rodrigues’ formula,

\[
T_n(x) = (-2)^n \frac{n!}{(2n)!} (1 - x^2)^{1/2} \frac{d^n}{dx^n} \left( [1 - x^2]^{n-1/2} \right).
\]

(a) Using the Rodrigues’ formula, show that, in the inner product \( \langle f, g \rangle = \int_{-1}^1 f(x)g(x)(1 - x^2)^{-1/2}dx \), \( T_n \) is orthogonal to all polynomials of degree \( n - 1 \) or less.

(b) Show that the generating function for the Chebyshev polynomials is

\[
\Phi(x, w) := \sum_{n=0}^\infty T_n(x)w^n = \frac{1 - xw}{1 - 2xw + w^2}.
\]

6. Do one of the following.

(a) State the Weierstrass approximation theorem and sketch a proof.

(b) Let \( \mathcal{H} \) be a complex, separable Hilbert space with inner product and norm given by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \). If \( L \) is a self-adjoint operator defined on a domain \( D \subseteq \mathcal{H} \), show that \( L \) has no residual spectrum and that its spectrum is real.
7. Consider the Schrödinger operator \( H u = -u'' + q(x)u \), with a compactly supported, continuous potential \( q(x) \geq 0 \). Show that the left and right transmission coefficients are equal; that is, \( T_L(k) = T_R(k) \).

8. Let \( L \) to be the self-adjoint operator \( L u = -u'' \), where \( D(L) = \{ u \in L^2(\mathbb{R}) : u'' \in L^2(\mathbb{R}) \} \).
   (a) Find the Green's function for \( L \).
   (b) Employ Stone's formula (i.e., the spectral theorem for self-adjoint operators) to obtain the Fourier transform,
      \[
      F(\mu) = \int_{-\infty}^{\infty} f(x)e^{i\mu x}dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\mu)e^{-i\mu x}d\mu.
      \]

9. Let \( \mathcal{H} \) be a complex (separable) Hilbert space, with \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) being the inner product and norm.
   (a) Let \( \lambda \in \mathbb{C} \) be fixed. If \( K : \mathcal{H} \to \mathcal{H} \) is a compact linear operator, show that the range of the operator \( L = I - \lambda K \) is closed.
   (b) Briefly explain why the operator \( K u(x) = \int_0^\pi \sin(x - t)u(t)dt \) is compact on \( L^2[0, \pi] \).
   (c) Determine the values of \( \lambda \) for which \( u = f + \lambda K u \) has a solution for all \( f \in \mathcal{H} \), given that \( K \) is the operator in 9b.