Mathematical Probability, Math 411, Homework 9

Remember the convolution formula for the sum of independent random variables $Z = X + Y$: $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$.

From the textbook solve the problems 1, 2, 3, 5, 6, 8, 11,14 at the end of the Chapter 4.

And also the problems below:

Problem 1. Let X be exponentially distributed with parameter λ. Find the PMF of $Y = \lceil X \rceil$, where $\lceil x \rceil$ for a real number x is the rounding of x to the nearest integer whose value is greater or equal to x. Identify the distribution of Y.

Problem 2. If X is uniform on the interval $[0, 2]$ find the PDF of $Y = X^2 - 2X$.

Problem 3. Let X be a uniform random variable in the interval $[a, b]$ and Y an exponential random variable with parameter λ. If X and Y are independent, compute the PDF of $Z = Y - X$.

Problem 4. Let X be a continuous random variable with the PDF f and the CDF F. Assume that $f(x) > 0$ for all real numbers x. Define the random variable $Y = F(X)$. Show that Y is uniform in the interval $[0, 1]$.

Problem 5. Give examples of (not independent) random variables X and Y, both of which are uniform in the interval $[0, 1]$ and such that

1) $\mathbf{P}(X + Y = 1) = 1$

2) $X + Y$ is uniform in the interval $[0, 2]$.

Problem 6. Two people have arranged to meet at a certain place and time. However, each of them is coming late and the person that comes first will have wait to the other person. The amount of time that the first and the second person are late is T_1 and T_2 respectively. Both T_1 and T_2 are exponential random variables with parameter λ and are independent. Find the PDF of the amount of time that the person arriving first will have to wait for the other person.

Problem 7. Let X be a continuous random variable with the PDF f and the CDF F. Assume that $f(x) > 0$ for all real numbers x. Define the random variable $Y = F(X)$. Show that Y is uniform in the interval $[0, 1]$.

(Hint: Since the derivative of the CDF F is the PDF f which is always strictly positive, F is monotonically increasing and has an inverse function h which is monotonically increasing. Now you can try to use the formula we derived with the help of the calculus formula for the derivative of the inverse function. Alternatively, you can show that $\mathbf{P}(Y \leq t) = t$ for $0 \leq t \leq 1$ with a little help of the inverse function h.)