BAD AND GOOD NEWS FOR STRASSEN'S LASER METHOD: BORDER RANK OF perm_3 AND STRICT SUBMULTIPLICATIVITY

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Abstract. We determine the border ranks of tensors that could potentially advance the known upper bound for the exponent ω of matrix multiplication. The Kronecker square of the small $q = 2$ Coppersmith-Winograd tensor equals the 3×3 permanent, and could potentially be used to show $\omega = 2$. We prove the negative result for complexity theory that its border rank is 16, resolving a longstanding problem. Regarding its $q = 4$ skew cousin in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$, which could potentially be used to prove $\omega \leq 2.11$, we show the border rank of its Kronecker square is at most 42, a remarkable sub-multiplicativity result, as the square of its border rank is 64. We also determine moduli spaces $\underline{VSP}$ for the small Coppersmith-Winograd tensors.

1. Introduction

This paper advances both upper and lower bound techniques in the study of the complexity of tensors and applies these advances to tensors that may be used to upper bound the exponent ω of matrix multiplication.

The exponent ω of matrix multiplication is defined as

$$\omega := \inf\{\tau \mid \text{ two } \mathbf{n} \times \mathbf{n} \text{ matrices may be multiplied using } O(\mathbf{n}^\tau) \text{ arithmetic operations}\}.$$

It is a fundamental constant governing the complexity of the basic operations in linear algebra. It is generally conjectured that $\omega = 2$. It has been known since 1988 that $\omega \leq 2.38$ [22] which was slightly improved upon 2011-2014 [49, 56, 38], and again in 2021 [3]. All new upper bounds on ω since 1987 have been obtained using Strassen's laser method, which bounds ω via auxiliary tensors, see any of [22, 7, 30] for a discussion. The bounds of 2.38 and below were obtained using the big Coppersmith-Winograd tensor as the auxiliary tensor. In [5] it was shown the big Coppersmith-Winograd tensor could not be used to prove $\omega < 2.3$ in the usual laser method.

In this paper we examine 6 tensors that potentially could be used to prove $\omega < 2.3$ with the laser method. Our approach is via algebraic geometry and representation theory, building on the recent advances in [11, 20]. We solve the longstanding problem (e.g., [7, Problem 9.8], [13, Rem. 15.44]) of determining the border rank of the Kronecker square of the only Coppersmith-Winograd tensor that could potentially prove $\omega = 2$ (the $q = 2$ small Coppersmith-Winograd tensor). The answer is a negative result for the advance of upper bounds, as it is 16, the maximum possible value. On the positive side, we show that a tensor that could potentially be used to prove $\omega < 2.11$ has border rank of its Kronecker square significantly smaller than the square of its border rank. While this result alone does not give a new upper bound on the exponent, it opens a promising new direction for upper bounds. We also develop new lower and upper bound techniques, and present directions for future research.

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The tensors we study are the small Coppersmith-Winograd tensor [21] $T_{cw,q}$ for $q = 2$ and its skew cousin [18] $T_{skewcw,q}$ for even $q \leq 10$ (five such). These tensors are defined for even $q > 10$ but they are only useful for the laser method when $q \leq 10$. The tensors $T_{cw,2}$ and $T_{skewcw,2}$ potentially could be used to prove $\omega = 2$. Explicitly, the small Coppersmith-Winograd tensors [22] are

$$T_{cw,q} = \sum_{j=1}^{q} a_0 \otimes b_j \otimes c_j + a_j \otimes b_0 \otimes c_j + a_j \otimes b_j \otimes c_0$$

and, for $q = 2p$ even, its skew cousins [18] are

$$T_{skewcw,q} = \sum_{\xi=1}^{p} a_0 \otimes b_\xi \otimes c_{\xi+p} - a_0 \otimes b_{\xi+p} \otimes c_\xi - a_\xi \otimes b_0 \otimes c_{\xi+p} + a_{\xi+p} \otimes b_0 \otimes c_\xi + a_\xi \otimes b_{\xi+p} \otimes c_0 - a_{\xi+p} \otimes b_\xi \otimes c_0.$$

The small Coppersmith-Winograd tensors are symmetric tensors and their skew cousins are skew-symmetric tensors. When $q = 2$, after a change of basis $T_{cw,2}$ is just a monomial written as a tensor, $T_{cw,2} = \sum_{\sigma \in \mathfrak{S}_3} a_{\sigma(1)} \otimes b_{\sigma(2)} \otimes c_{\sigma(3)}$ and $T_{skewcw,2} = \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma) a_{\sigma(1)} \otimes b_{\sigma(2)} \otimes c_{\sigma(3)}$. Here $\mathfrak{S}_3$ denotes the permutation group on three elements.

We need the following definitions to state our results:

A tensor $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ has *rank one* if $T = a \otimes b \otimes c$ for some $a \in A$, $b \in B$, $c \in C$, and the *rank* of T, denoted $\mathbf{R}(T)$, is the smallest r such that T may be written as a sum of r rank one tensors. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T may be written as a limit of rank r tensors. In geometric language, the border rank is smallest r such that $[T] \in \sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$, where $\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ denotes the r-th secant variety of the Segre variety of rank one tensors.

For symmetric tensors $T \in S^3 A \subset A \otimes A \otimes A$ we may also consider the *Waring* or *symmetric rank* of T, $\mathbf{R}_S(T)$, the smallest r such that $T = \sum_{s=1}^{r} v_s \otimes v_s \otimes v_s$ for some $v_s \in A$, and the Waring border rank $\underline{\mathbf{R}}_S(T)$, the smallest r such that T may be written as a limit of Waring rank r symmetric tensors.

For tensors $T \in A \otimes B \otimes C$ and $T' \in A' \otimes B' \otimes C'$, the *Kronecker product* of T and T' is the tensor $T \boxtimes T' := T \otimes T' \in (A \otimes A') \otimes (B \otimes B') \otimes (C \otimes C')$, regarded as 3-way tensor. Given $T \in A \otimes B \otimes C$, the *Kronecker powers* of T are $T^{\boxtimes N} \in A^{\otimes N} \otimes B^{\otimes N} \otimes C^{\otimes N}$, defined iteratively. Rank and border rank are submultiplicative under Kronecker product: $\mathbf{R}(T \boxtimes T') \leq \mathbf{R}(T)\mathbf{R}(T')$, $\underline{\mathbf{R}}(T \boxtimes T') \leq \underline{\mathbf{R}}(T)\underline{\mathbf{R}}(T')$, and both inequalities may be strict.

Strassen's *laser method* [51, 21] obtains upper bounds on ω by showing a random degeneration of a large Kronecker power of a simple tensor degenerates further to a sum of disjoint matrix multiplication tensors, and then applying Schönhage's asymptotic sum inequality [44]. The relevant results for this paper are:

For all k and q, [22]

$$\omega \leq \log_q \left(\frac{4}{27} (\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k}))^{\frac{3}{k}} \right). \tag{1}$$

For all k and even q, [18]

$$\omega \leq \log_q \left(\frac{4}{27} (\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes k}))^{\frac{3}{k}} \right). \tag{2}$$

Coppersmith-Winograd [22] showed $\underline{\mathbf{R}}(T_{cw,q}) = q + 2$. Applied to (1) with $k = 1$ and $q = 8$ gives $\omega \leq 2.41$, which was the previous record before 2.38.

The most natural way to upper bound the exponent of matrix multiplication would be to upper bound the border rank of the matrix multiplication tensor directly. There are very few results in this direction: work of Strassen [53], Bini [6], Pan (see, e.g., [42]), and Smirnov (see, e.g., [47]) are all we are aware of. In order to lower the exponent further with the matrix multiplication tensor the first opportunity to do so would be to show the border rank of the 6×6 matrix multiplication tensor equaled its known lower bound of 69 from [36].

The only still viable proposed paths to prove $\omega < 2.3$ that we are aware of would be to obtain border rank upper bounds for a Kronecker power of a small ($q \leq 10$) Coppersmith-Winograd tensor (this path has been proposed since 1989) or its skew cousin (more recently proposed in [18]). The results in this paper take a few steps further on these two paths. There is no proposed path that we are aware of to prove $\omega > 2.3$ other than by proving border rank lower bounds for the matrix multiplication tensor (or its symmetrized or skew-symmetrized versions [14]) for all n.

1.1. **Main Results.**

After the barriers of [5], the auxiliary tensor viewed as most promising for upper bounding the exponent, or even proving it is two, is the small Coppersmith-Winograd tensor, or more precisely its Kronecker powers. In [18] bad news in this direction was shown for the square of most of these tensors and even the cube. Left open was the square of $T_{cw,2}$ as it was unaccessible by the technology available at the time (Koszul flattenings and the border substitution method), although it was shown that $15 \leq \underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2}) \leq 16$. With the advent of border apolarity [11, 20] and the Flag Condition (Proposition 2.5) that strengthens it, we are able to resolve this last open case. See Remark 5.4 for an explanation why this result was previously unaccessible, even with the techniques of [11, 20]. The result for the exponent is negative:

Theorem 1.1. $\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2}) = 16$.

For a detailed discussion of the relation of border rank bounds to the exponent for Kronecker powers of the small Coppersmith-Wingrad tensor and its skew cousin, see Section 1 of [18].

In [18] it was observed that $T_{cw,2}^{\boxtimes 2} = \mathrm{perm}_3$, the 3×3 permanent considered as a tensor. Y. Shitov [45] has shown that the Waring rank of perm_3 is at least 16, which matches the upper bound of [27].

Remark 1.2. P. Comon [9] had conjectured that for symmetric tensors their Waring rank equals their tensor rank and it has similarly be conjectured that their Waring border rank equals their tensor border rank. While Comon's conjecture was shown to be false in general by Shitov [46], Theorem 1.1 shows that both versions hold for perm_3.

Theorem 1.1 is proved in §5.

We determine the border rank of $T_{skewcw,q}$ in the range relevant for the laser method:

Theorem 1.3. $\underline{\mathbf{R}}(T_{skewcw,q}) \leq \frac{3}{2}q + 2$ *and equality holds for* $q \leq 10$.

While this is less promising than the equality $\underline{\mathbf{R}}(T_{cw,q}) = q + 2$, in [18] a significant drop in the border rank of $T_{skewcw,2}^{\boxtimes 2} = \det_3$ was shown, namely that it is 17 rather than $25 = \underline{\mathbf{R}}(T_{skewcw,2})^2$. (The upper bound was shown in [18] and the lower bound in [20].) Theorem 1.3 implies $\underline{\mathbf{R}}(T_{skewcw,4})^2 = 64$. The following theorem is the largest drop in border rank under a Kronecker square that we are aware of:

Theorem 1.4. $(*)$ $\underline{\mathbf{R}}(T_{skewcw,4}^{\boxtimes 2}) \leq \underline{\mathbf{R}}_S(T_{skewcw,4}^{\boxtimes 2}) \leq 42$.

The Theorem is marked with a $(*)$ because the result is only shown to hold numerically. The expression we give has largest error 4.4×10^{-15}. While we could have presented a solution to higher accuracy, we were unable to find an exact expression. The new numerical techniques used to obtain this decomposition are described in §9. We also give a much simpler Waring border rank 17 expression for $\det_3 = T_{skewcw,2}^{\boxtimes 2}$ than the one in [18], see §8.

Using Koszul flattenings (see §4) we show $\underline{\mathbf{R}}(T_{skewcw,4}^{\boxtimes 2}) \geq 39$. For the cube we show $\underline{\mathbf{R}}(T_{skewcw,4}^{\boxtimes 3}) \geq 219$ whereas for its cousin we have $180 \leq \underline{\mathbf{R}}(T_{cw,4}^{\boxtimes 3}) \leq 216$. We also prove, using Koszul flattenings, lower bounds for $\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes 2})$ and $\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes 3})$ for $q \leq 10$. These results are all part of Theorem 4.1.

Remark 1.5. Starting with the fourth Kronecker power it is possible the border rank of $T_{skewcw,q}^{\boxtimes 4}$ is less than that of $T_{cw,q}^{\boxtimes 4}$, for $q \in \{2,6,8\}$. The best possible upper bound on ω obtained from some $T_{skewcw,q}^{\boxtimes 4}$ would be $\omega \leq 2.39001322$ which could potentially be attained with $q = 6$. Starting with the fifth Kronecker power it is potentially possible to beat the current world record for ω with $T_{skewcw,q}$ and for $T_{cw,q}$ it is already possible with the fourth power.

Strassen's asymptotic rank conjecture [52] posits that for all concise tensors $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ (see Definition 1.8) with regular positive dimensional symmetry group (called *tight tensors*), $\lim_{k\to\infty}[\underline{\mathbf{R}}(T^{\boxtimes k})]^{\frac{1}{k}} = m$. As a first step towards this conjecture it is an important problem to determine which tensors T satisfy $\underline{\mathbf{R}}(T^{\boxtimes 2}) < \underline{\mathbf{R}}(T)^2$. We discuss what we understand about this problem in §3.2.

A variety that parametrizes all possible (minimal) border rank decompositions of a given tensor T, denoted $\underline{VSP}(T)$, is defined in [11]. This variety naturally sits in a product of Grassmannians, see §3.1 for the definition. We observe that in many examples $\underline{VSP}(T)$ often has a large dimension when $\underline{\mathbf{R}}(T^{\boxtimes 2}) < \underline{\mathbf{R}}(T)^2$ (although not always), and in all examples we know of, when $\underline{VSP}(T)$ is zero-dimensional one also has $\underline{\mathbf{R}}(T^{\boxtimes 2}) = \underline{\mathbf{R}}(T)^2$. This is reflected in the following results:

Theorem 1.6. *For $q > 2$, $\underline{VSP}(T_{cw,q})$ is a single point.*

Theorem 1.7. *$\underline{VSP}(T_{cw,2})$ consists of three points.*

More precise versions of these results and their proofs are given in §6.

In contrast $\underline{VSP}(T_{skewcw,q})$ is positive dimensional, at least for all q relevant for complexity theory ($q \leq 10$). Explicitly, $\underline{VSP}(T_{skewcw,2})$ is at least 8-dimensional, see Corollary 3.2, and for $4 \leq q \leq 10$, $\dim \underline{VSP}(T_{skewcw,q}) \geq \binom{q/2}{2}$, see Corollary 7.1.

Border apolarity is just in its infancy. In §2.1 we give a history leading up to it. In §2.2 we explain results from border apolarity needed in this paper. In §2.3 we discuss challenges to getting better results with the method and take first steps to overcome them in §2.4. In particular, Proposition 2.5 was critical to the proof of Theorem 1.1 as it enables one to substantially reduce the border apolarity search space in certain situations (weights occurring with multiplicities).

1.2. **Previous border rank bounds on $T_{cw,q}^{\boxtimes k}$ and $T_{skewcw,q}^{\boxtimes k}$.**

- $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 2}) = (q+2)^2$ for $q > 2$ and $15 \leq \underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2}) \leq 16$. [18]
- $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 3}) = (q+2)^3$ for $q > 4$. [18]

- $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 2}) = 17$. [20]
- $\underline{\mathbf{R}}(T_{skewcw,q}) \geq q+3$. [18]
- For all $q > 4$ and all k, $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k}) \geq (q+2)^3(q+1)^{k-3}$ and $\underline{\mathbf{R}}(T_{cw,4}^{\boxtimes k}) \geq 36 \cdot 5^{k-2}$. [18]
- $\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes k}) \geq 15 \cdot 3^k$. [18]

With the exception of the proof $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 2}) \geq 17$, which was obtained via border apolarity, these lower bounds were obtained using Koszul flattenings.

Previous to these it was shown that $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes k}) \geq (q+1)^k + 2^k - 1$ using the border substitution method [8].

1.3. **Definitions/Notation.** Throughout, A, B, C will denote complex vector spaces of dimension m. We let $\{a_i\}$ denote a basis of A, with either $0 \leq i \leq m-1$ or $1 \leq i \leq m$ and similarly for $\{b_j\}$ and $\{c_k\}$. The dual space to A is denoted A^*. Since our vector spaces have names, we re-order them freely without danger of confusion. The $\mathbb{Z}$-graded algebra of symmetric tensors is denoted $Sym(A) = \oplus_d S^d A$, it is also the algebra of homogeneous polynomials on A^*. For $X \subset A$, $X^\perp := \{\alpha \in A^* \mid \alpha(x) = 0 \forall x \in X\}$ is its annihilator, and $\langle X \rangle \subset A$ denotes the span of X. Projective space is $\mathbb{P}A = (A \backslash \{0\})/\mathbb{C}^*$, and if $x \in A\backslash\{0\}$, we let $[x] \in \mathbb{P}A$ denote the associated point in projective space (the line through x). The general linear group of invertible linear maps $A \to A$ is denoted $GL(A)$ and the special linear group of determinant one linear maps is denoted $SL(A)$. The permutation group on r elements is denoted $\mathfrak{S}_r$.

The Young diagram associated to a partition $(p_1, \cdots, p_d)$ is an array of left-aligned boxes with p_j boxes in the j-th row.

The Grassmannian of r-planes through the origin is denoted $G(r, A)$, which we will view in its Plücker embedding $G(r, A) \subset \mathbb{P}\Lambda^r A$. We let $Gr(r, A)$ denote the Grassmannian of *codimension* r planes.

For a set $Z \subset \mathbb{P}A$, $\overline{Z} \subset \mathbb{P}A$ denotes its Zariski closure, $\widehat{Z} \subset A$ denotes the cone over Z union the origin, $I(Z) = I(\widehat{Z}) \subset Sym(A^*)$ denotes the ideal of Z, and $\mathbb{C}[\widehat{Z}] = Sym(A^*)/I(Z)$, denotes the homogeneous coordinate ring of $\widehat{Z}$. Both $I(Z)$, $\mathbb{C}[\widehat{Z}]$ are $\mathbb{Z}$-graded by degree.

We will be dealing with ideals on products of three projective spaces, that is, we will be dealing with polynomials that are homogeneous in three sets of variables, so our ideals with be $\mathbb{Z}^{\oplus 3}$-graded. More precisely, we will study ideals $I \subset Sym(A^*) \otimes Sym(B^*) \otimes Sym(C^*)$, and I_{stu} denotes the component in $S^s A^* \otimes S^t B^* \otimes S^u C^*$.

For $T \in A \otimes B \otimes C$, define the *symmetry group* of T, $G_T := \{g = (g_1, g_2, g_3) \in GL(A) \times GL(B) \times GL(C) \mid g \cdot T = T\}$.

Given $T, T' \in A \otimes B \otimes C$, we say that T *degenerates* to T' if $T' \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$, the closure of the orbit of T. The closures are the same in the Euclidean and Zariski topologies.

Definition 1.8. Given $T \in A \otimes B \otimes C$, we may consider it as a linear map $T_C : C^* \to A \otimes B$, and we let $T(C^*) \subset A \otimes B$ denote its image, and similarly for permuted statements. A tensor T is *A-concise* if the map T_A is injective, i.e., if it requires all basis vectors in A to write down in any basis, and T is *concise* if it is A, B, and C concise. A tensor is *1_A-generic* if $T(A^*) \subset B \otimes C$ contains an element of maximal rank m.

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2. Border apolarity and the challenges it faces

2.1. **History.** Until very recently, essentially the only way to prove border rank lower bounds for a tensor T was to find a polynomial P in the ideal of $\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))$ such that $P(T)\neq 0$. (See [40] for an exception.) The first nontrivial equations for tensors were found by Strassen in 1983 [50], although the equations essentially date back to 1877 when E. Toeplitz [55] wrote the equations in the partially symmetric case. No further equations were found until 2004 [31], then 2008 [32], then 2013 [35, 37], and these are the state of the art. The equations (and a much broader class of equations) are known to have limits (see, e.g., [24]), essentially one could not prove border rank lower bounds better than $2m-3$ for tensors in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$. A small way to improve upon this was developed in [34, 8]: this *border substitution method*, which generalizes the classical substitution method to prove rank lower bounds, is only applicable in practice to tensors with positive dimensional symmetry groups: Let $T\in A\otimes B\otimes C$ be A-concise. Let G_T be the symmetry group of T and let $\mathbb{B}_T\subset G_T$ be a Borel subgroup. Let $Gr(t,A^*)$ denote the Grassmannian of *codimension* t-planes in A^*. Note that $\mathbb{B}_T$ acts on $Gr(t,A^*)$ so it makes sense to discuss its Borel fixed elements. Then

$$\underline{\mathbf{R}}(T)\geq \min_{A'\in Gr(t,A^*),\textbf{Borel fixed}}\underline{\mathbf{R}}(T|_{A'\otimes B^*\otimes C^*})+t. \tag{3}$$

This enables one to prove border rank lower bounds on T by proving border rank lower bounds via known equations on the restrictions of T to all Borel fixed elements of the Grassmannian $Gr(t,A^*)$. In [11] Buczynska-Buczynski introduced *Border apolarity*, which generalizes the classical apolarity for rank to border rank, and $\overline{VSP}$ which generalizes the the Variety of Sums of Powers (VSP, see, e.g., [43]) for rank decompositions to border rank decompositions.

2.2. **Border apolarity.** If one has a border rank decomposition $T=\lim_{\epsilon\to 0}\sum_{j=1}^r T_j(\epsilon)$, for each $\epsilon>0$, one obtains an ideal of polynomials in the coordinate ring of the Segre $Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$ vanishing on the r points $[T_1(\epsilon)]\sqcup\cdots\sqcup[T_r(\epsilon)]$. These are ideals in three sets of variables (those of A,B,C), and since border rank decompositions only utilize a finite number of terms in the Taylor expansion of the $T_j(\epsilon)$, one may assume that for all $\epsilon>0$, the r points are in general position by modifying the higher order terms in the series. This has the effect that in each multi-degree $I_{stu,\epsilon}\subset S^sA^*\otimes S^tB^*\otimes S^uC^*$ has codimension r whenever $s+t+u>1$. Thus for each s,t,u there is a limiting $I_{stu}\in Gr(r,S^sA^*\otimes S^tB^*\otimes S^uC^*)$. Moreover, generalizing (3), one may assume that each of the I_{stu} is $\mathbb{B}_T$-fixed. By the construction in [26] these limiting spaces fit together to form an ideal. In particular the ideal annihilates T, which in practice means $I_{110}\subseteq T(C^*)^\perp$, $I_{101}\subseteq T(B^*)^\perp$, $I_{011}\subseteq T(A^*)^\perp$ and $I_{111}\subset T^\perp$. Moreover, since ideals are closed under multiplication, the image of the direct sum of the three multiplication maps

$$I_{s-1,t,u}\otimes A^*\oplus I_{s,t-1,u}\otimes B^*\oplus I_{s,t,u-1}\otimes C^*\to S^sA^*\otimes S^tB^*\otimes S^uC^*,$$

must be contained in I_{stu}. In particular the image must have codimension at least r, which translates to rank conditions on the map. Call the map the (stu)-map and the rank condition the (stu)-test.

Write $E_{stu}=I_{stu}^\perp$. It will be convenient to phrase the codimension tests dually:

Proposition 2.1. [20, Prop. 3.1] *The* (210)*-test is passed if and only if the skew-symmetrization map*

$$A \otimes E_{110} \to \Lambda^2 A \otimes B \tag{4}$$

has kernel of dimension at least r*. The kernel is* $(A \otimes E_{110}) \cap (S^2 A \otimes B)$*.*

The (stu)*-test is passed if and only if the triple intersection*

$$(E_{s,t,u-1} \otimes C) \cap (E_{s,t-1,u} \otimes B) \cap (E_{s-1,t,u} \otimes A) \tag{5}$$

has dimension at least r*.*

We will make repeated use of the following lemma:

Lemma 2.2 (Fixed ideal Lemma [11])**.** *If* T *has symmetry group* G_T*, then if there exists an ideal as above then there exists one that is fixed under the action of a Borel subgroup of* G_T *which we will denote* $\mathbb{B}_T$*. In particular, if* G_T *contains a torus, if there exists such an ideal, then there exists one fixed under the action of the torus.*

Border apolarity provides both lower bounds and a guide to proving upper bounds. For example, the (111) space for $T_{skewcw,q}$ described in the proof of Theorem 1.3 hints at the formula (21), where the terms linear in t appear in the (111) space.

2.3. **Challenges facing border apolarity.**

In modern algebraic geometry the study of geometric objects (algebraic varieties) is replaced by the study of the ideal of polynomials that vanish on a variety. The study of a set of r points $\{z_1, \cdots, z_r\}$ in affine space $\mathbb{C}^N$ is replaced by the study of its ideal, more precisely the quotient $\mathbb{C}[x_1, \cdots, x_N]/I_{z_1 \sqcup \cdots \sqcup z_r}$ where $\mathbb{C}[x_1, \cdots, x_N]$ is the ideal of all polynomials on $\mathbb{C}^N$ and $I_{z_1 \sqcup \cdots \sqcup z_r}$ is the ideal. Note that ring $\mathbb{C}[x_1, \cdots, x_N]/I_{z_1 \sqcup \cdots \sqcup z_r}$ is a vector space of dimension r, called the coordinate ring of the variety. (In our case we will be concerned with r points on the Segre variety $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ but the issues about to be discussed are local and there is no danger working in affine space.) The study becomes one of such rings, and one no longer requires them to correspond to ideals of points, only that the vector space has dimension r and that the ideal is saturated. Such ideals are called *zero dimensional schemes of length* r. If the ideal corresponds to r distinct points one says the scheme is *smooth*. A central challenge of border apolarity as a tool in the study of border rank, is that applied naïvely, it only determines necessary conditions for an ideal to be the limit of a sequence of such ideals. One could split the problem of detecting non-border rank ideals into two: first, just get rid of the ideals that are not limits of ideals of zero dimensional schemes, then, given an ideal that is a limit of ideals of zero dimensional schemes of length r, determine if it is a limit of ideals of smooth schemes (*smoothability* conditions). In this paper we address the first problem and the new additional necessary conditions we obtain (Proposition 2.5) are enough to enable us to determine $\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes 2})$ via border apolarity. In §2.6 we show that ideals that fail to deform to saturated ideals occur already for quite low border rank. The second problem is ongoing work with J. Buczyński and his group in Warsaw.

The second problem is a serious issue: The *cactus rank* [11, 10] of a tensor T is the smallest r such that T lies in the span of a zero dimensional scheme of length r supported on the Segre variety. The cactus border rank of T, $\underline{\mathbf{CR}}(T)$ is the smallest r such that T is a limit of tensors of cactus rank r. One has $\underline{\mathbf{R}}(T) \geq \underline{\mathbf{CR}}(T)$ and for almost all tensors the inequality is strict. The (stu) tests are tests for cactus border rank. Cactus border rank is not known to be relevant for complexity theory, thus the failure of current border apolarity technology to distinguish between them is a barrier to future progress. Moreover, the cactus variety fills the ambient space of $\mathbb{P}(\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m)$ at latest border rank $6m-4$, see [25, Ex. 6.2 case $k=3$].

2.4. **Viability and the flag conditions.** We begin in the general context of secant varieties with a preliminary observation:

For a projective variety $X\subset\mathbb{P}^N$, define its variety of secant $\mathbb{P}^{r-1}$'s,

$$\sigma_r(X):=\overline{\bigcup_{x_1,\cdots,x_r\in X}\langle x_1,\cdots,x_r\rangle}.$$

Proposition 2.3. *Let $X\subset\mathbb{P}V$ be a projective variety and let $\mathbb{P}E\subset\sigma_r(X)$ be a $\mathbb{P}^{r-1}$ arising from a border rank r decomposition of a point on $\sigma_r(X)$. Then there exists a complete flag $E_1\subset E_2\subset\cdots\subset E_r=E$ such that for all $1\le j\le r$, $\mathbb{P}E_j\subset\sigma_j(X)$.*

Proof. We may write $E=\lim_{t\to0}\langle x_1(t),\cdots,x_r(t)\rangle$ where $x_j(t)\in X$ and the limit is taken in the Grassmannian $G(r,V)$ (in particular, for all $t\neq0$ we may assume $x_1(t),\cdots,x_r(t)$ are linearly independent). Then take $E_j=\lim_{t\to0}\langle x_1(t),\cdots,x_j(t)\rangle$ where the limit is taken in the Grassmannian $G(j,V)$. □

Let $T\in A\otimes B\otimes C$ and let E_{stu} be an r-dimensional space that is $I_{stu}^{\perp}$ for a multi-graded ideal that passes all border apolarity tests up to total degree $s+t+u+1$.

Definition 2.4. An multi-graded ideal, or an E_{stu}, associated to a potential border rank decomposition of T is *viable* if it arises from an actual border rank decomposition.

Viability implies $\mathbb{P}E_{stu}\subset\sigma_r(Seg(v_s(\mathbb{P}A)\times v_t(\mathbb{P}B)\times v_u(\mathbb{P}C)))$. Here $v_s:\mathbb{P}A\to\mathbb{P}(S^sA)$ is the Veronese re-embedding, $v_s([a])=[a^s]$.

To a $\mathbf{c}$-dimensional subspace $E\subset A\otimes B$, one may associate a tensor $T\in A\otimes B\otimes\mathbb{C}^{\mathbf{c}}$, well-defined up to isomorphism, such that $T(\mathbb{C}^{\mathbf{c}*})=E$. Much of the lower bound literature exploits this correspondence to reduce questions about tensors to questions about linear subspaces of spaces of matrices. (This idea appears already in [50].) The following proposition exploits this dictionary to obtain new conditions for viability of candidate E_{stu}'s:

Proposition 2.5. *[Flag conditions] If E_{110} is viable, then there exists a $\mathbb{B}_T$-fixed filtration of E_{110}, $F_1\subset F_2\subset\cdots\subset F_r=E_{110}$, such that $F_j\subset\sigma_j(Seg(\mathbb{P}A\times\mathbb{P}B))$. Let $T_j\in A\otimes B\otimes\mathbb{C}^j$ be a tensor equivalent to the subspace F_j. Then $\underline{\mathbf{R}}(T_j)\le j$.*

Similarly, if E_{stu} is viable, there are complete flags in E_{stu},A,B,C such that for all $j<m$, $E_{stu,j}\subset S^sA_j\otimes S^tB_j\otimes S^uC_j$ and for all $j\le r$, $\mathbb{P}E_{stu,j}\subset\sigma_j(Seg(v_s(\mathbb{P}A)\times v_t(\mathbb{P}B)\times v_u(\mathbb{P}C)))$.

Proof. Set $\widehat{C}=C\oplus\mathbb{C}^{r-m}$. Then there exists $\widehat{T}\in A\otimes B\otimes\widehat{C}$ such that $\widehat{T}(\widehat{C}^*)=E_{110}$ and $\underline{\mathbf{R}}(\widehat{T})\le r$. In this case the flag condition [33, Cor. 2.3] implies that since $\widehat{T}\in A\otimes B\otimes\widehat{C}=\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^r$ with $r\ge m$ is concise of minimal border rank r, there exists a complete flag $C_1\subset C_2\subset\cdots\subset C_r=\widehat{C}^*$ such that $\widehat{T}(C_k)\subset\sigma_k(Seg(\mathbb{P}A\times\mathbb{P}B))$. Take $F_k=\widehat{T}(C_k)$. The proof that the flag may be taken to be Borel fixed is the same as in the Fixed ideal lemma.

The second assertion follows from the preceding discussion. □

Proposition 2.5 provides additional conditions E_{stu} must satisfy for viability beyond the border apolarity tests. It allows one to utilize the known conditions for minimal border rank in a non-minimal border rank setting.

When T_j is concise, Proposition 2.5 is quite useful as there are many known conditions for concise tensors to be of minimal border rank. In particular it must have symmetry Lie algebra of dimension at least $2j-2$ and if it is $1_{\mathbb{C}^j}$-generic (for any of the factors), it must satisfy the End-closed condition (see [33]).

Remark 2.6. Proposition 2.5 also applies to cactus border rank decompositions, so it is a "non-deformable to saturated" removal condition rather than a smoothability one.

By the classification of tensors of border rank at most three [12, Thm. 1.2(iv)] the possibilities for the first two filtrands of E_{110} are $F_1 = \langle a\otimes b\rangle$, $F_{2a} = \langle a\otimes b, a'\otimes b'\rangle$ or $F_{2b} = \langle a\otimes b, a\otimes b' + a'\otimes b\rangle$ corresponding to either two distinct rank one points or a rank one point and a tangent vector, and there are five possibilities for F_3:

1. $F_{3aa} = \langle a\otimes b, a'\otimes b', a''\otimes b''\rangle$ (three distinct points)
2. $F_{3aab} = \langle a\otimes b, a\otimes b' + a'\otimes b, a''\otimes b''\rangle$ (two points plus a tangent vector to one of them)
3. $F_{3bc} = \langle a\otimes b, a\otimes b' + a'\otimes b, a''\otimes b + a'\otimes b' + a\otimes b''\rangle$ (points of the form $x(0), x'(0), x''(0)$ for a curve $x(t) \subset Seg(\mathbb{P}A \times \mathbb{P}B)$)
4. $F_{3abb} = \langle a\otimes b, a\otimes b' + a'\otimes b, a\otimes b'' + a''\otimes b\rangle$ (point plus two tangent vectors)
5. $F_{3bd} = \langle a\otimes b, a\otimes b', a'\otimes b + a''\otimes b' + a\otimes b''\rangle$ (sum of tangent vectors to two colinear points $x' + y'$) or its mirror $F_{3bd} = \langle a\otimes b, a'\otimes b, a\otimes b' + a'\otimes b'' + a''\otimes b\rangle$.

The space E_{110} contains the distinguished subspace $T(C^*)$. Write E'_{110} for a choice of a complement to $T(C^*)$ in E_{110}.

Corollary 2.7. *If E_{110} is viable and $\mathbb{P}T(C^*) \cap \sigma_k(Seg(\mathbb{P}A \times \mathbb{P}B)) = \emptyset$, then there exists a choice of E'_{110} such that $F_k \subset E'_{110}$.*

Proof. Say otherwise, then there exists $M \in F_k \cap T(C^*)$. This contradicts $T(C^*) \cap \sigma_k(Seg(\mathbb{P}A \times \mathbb{P}B)) = \emptyset$. □

The following Corollary originally appeared in [8, Cor. 4.2]:

Corollary 2.8. *If $\mathbb{P}T(C^*) \cap \sigma_q(Seg(\mathbb{P}A \times \mathbb{P}B)) = \emptyset$, then $\underline{\mathbf{R}}(T) \geq m + q$.*

Although we have stronger lower bounds, Corollary 2.8 provides the following "for free":

Corollary 2.9. *For all k, $\underline{\mathbf{R}}(T_{cw,2}^{\boxtimes k}) \geq 3^k + 2^k - 1$ and $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes k}) \geq 3^k + 2^k - 1$.*

The first assertion originally appeared in [8].

Proof. Let $i_\alpha, j_\beta \in \{1,2,3\}$. Then

$$T_{cw,2}^{\boxtimes k}(C^*) = \langle \sum_{\sigma \in \mathbb{Z}_2^k} \sigma \cdot (a_{i_1,\cdots,i_k} \otimes b_{j_1,\cdots,j_k}) \mid i_\alpha \neq j_\alpha \forall 1 \leq \alpha \leq k\rangle$$

and the action of σ is by swapping indices. This transparently is of rank bounded below by 2^k. The case of $T_{skewcw,2}^{\boxtimes k}$ is the same except that the coefficients appear with signs. □

2.5. **Free, pure and mixed kernels.** Define three types of contribution to the kernel of the (210)-map: the *free* kernel

$$\kappa_f := \dim[(T(C^*)\otimes A)\cap(S^2A\otimes B)],$$

the *pure* kernel

$$\kappa_p = \min_{E'_{110}}\dim[(A\otimes E'_{110})\cap(S^2A\otimes B)],$$

where the min is over all choices of $E'_{110}\subset E_{110}$, and the *mixed* kernel

$$\kappa_m = \dim[(A\otimes E_{110})\cap(S^2A\otimes B)] - \kappa_p - \kappa_f$$

corresponding to elements of the kernel arising from linear combinations of elements of $A\otimes E'_{110}$ and $A\otimes T(C^*)$. In this language, E'_{110} passes the (210) test if and only if $\kappa_p+\kappa_m\geq r-\kappa_f$. Define corresponding κ'_p,κ'_m for the (120)-test.

Conjecture 2.10. *If $r>m$ and E_{110} is such that $\kappa_m,\kappa'_m=0$, then it is not viable.*

Intuitively, if E'_{110} never "sees" the tensor, it should not be viable.

2.6. **Limitations of the total degree 3 border apolarity tests.**

Proposition 2.11. *Let $m\geq 9$, but $m\neq 10,15$. Then for any tensor in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$, there are candidate ideals passing all degree three tests for border rank at most r when $r\geq 2m$.*

More generally, setting $r=m+k^2$, there are candidate ideals in total degree two passing all degree three tests once $m\leq\frac{k^3}{2}-\frac{k^2}{2}$. In particular, for all $\epsilon>0$, $r\geq m+m^{\frac{1}{3}+\epsilon}$, and m sufficiently large, there are such candidate ideals.

Proof. For the first assertion, it suffices to prove the case $r=2m$ and the tensor T is concise. Set $k=\lfloor\sqrt{m}\rfloor$, $t=k+\lceil\frac{m-k^2}{2}\rceil$, and $t'=k+\lfloor\frac{m-k^2}{2}\rfloor$. Take $E'_{110}=\langle a_1,\cdots,a_k\rangle\otimes\langle b_1,\cdots,b_k\rangle+\langle a_{k+1},\cdots,a_{t'}\rangle\otimes b_1+a_1\otimes\langle b_{k+1},\cdots,b_t\rangle$ and similarly for the other spaces. Then

$$(E_{110}\otimes A)\cap(S^2A\otimes B)\supseteq$$
$$S^2\langle a_1,\cdots,a_k\rangle\otimes\langle b_1,\cdots,b_k\rangle\oplus\langle a_{k+1},\cdots,a_{t'}\rangle\cdot\langle a_1,\cdots,a_k\rangle\otimes b_1\oplus S^2\langle a_{k+1},\cdots,a_{t'}\rangle\otimes b_1+a_1^{\otimes 2}\otimes\langle b_{k+1},\cdots,b_t\rangle.$$

This has dimension $\binom{k+1}{2}k+(t'-k)k+\binom{t'-k+1}{2}+(t-k)$ which is at least $2m$ in the specified range. (The only value greater than 8 the inequality fails for is $m=10$.) Similarly the (120) test is passed at least as easily. Finally

$$(E_{110}\otimes C)\cap(E_{101}\otimes B)\cap(E_{011}\otimes A)\supseteq\langle a_1,\cdots,a_k\rangle\otimes\langle b_1,\cdots,b_k\rangle\otimes\langle c_1,\cdots,c_k\rangle\oplus\langle T\rangle$$

which has dimension k^3+1 which is at least $2m$ in the range of the proposition. (The only value greater than 8 the inequality fails for is $m=15$.)

The second assertion follows with the same E'_{110}, taking $r=m+k^2$ and $t,t'=0$. $\square$

Example 2.12. For $T_{cw,2}^{\boxtimes 3}$ it is easy to get E'_{110} of dimension 21 (so for border rank $48<63$) that pass the (210) and (120) tests. Take E'_{110} spanned by rank one basis vectors such that the associated Young diagram is a staircase. Then $\kappa_p=\kappa'_p=1(6)+2(5)+3(4)+4(3)+5(2)+6(1)=56>48$.

3. Moduli and submultiplicativity

3.1. **Moduli spaces $\underline{VSP}$.** Following [11], define $\underline{VSP}(T)$ to be the set of ideals as in §2.2 arising from a border rank $\underline{\mathbf{R}}(T)$ decomposition of T. (In the notation of [11] this is $\underline{VSP}(T, \underline{\mathbf{R}}(T))$.) Since for zero dimensional schemes of a fixed length (and more generally for schemes with a fixed Hilbert polynomial), there is a uniform bound on degrees of generators of their ideals, this is a finite dimensional variety which naturally embedds in a product of Grassmannians.

A more classical object also of interest is $\underline{VSP}_{A\otimes B\otimes C}(T) \subset G(\underline{\mathbf{R}}(T), A\otimes B\otimes C)$, which just records the $\underline{\mathbf{R}}(T)$-planes giving rise to a border rank decomposition, i.e., the annihilator of the (111)-component of the ideal. In particular $\dim(\underline{VSP}_{A\otimes B\otimes C}(T)) \leq \dim(\underline{VSP}(T))$.

It will be useful to state the following result in a more general context: Let $X \subset \mathbb{P}V$ be a variety not contained in a hyperplane, assume $\sigma_{r-1}(X) \neq \mathbb{P}V$ and write $\dim \sigma_r(X) = r\dim(X) + r - 1 - \delta$. Consider the incidence correspondence

$$S_r(X) = \overline{\{((x_1, \cdots, x_r), y, V) \in X^{\times r} \times \mathbb{P}V \times G(r, V) \mid y \in \langle x_1, \cdots, x_r\rangle \subseteq V\}},$$

and its projection maps

$$\begin{array}{ccc} & S_r(X) & \\ \swarrow & & \searrow \\ G(r,V) & & \sigma_r(X). \end{array}$$

Call the projections π_G, π_σ. We have $\dim S_r(X) = r\dim(X) + r - 1$ so for $y \in \sigma_r(X)_{general}$, $\dim(\pi_\sigma^{-1}(y)) = \delta$.

Define $\underline{VSP}_{X,\mathbb{P}V}(y) := \pi_G \pi_\sigma^{-1}(y)$. When $X = Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$, $y = T$, and $V = A\otimes B\otimes C$, this is $\underline{VSP}_{A\otimes B\otimes C}(T)$.

Proposition 3.1. *For all $y \in \sigma_r(X)$, $\dim \underline{VSP}_{X,\mathbb{P}V}(y) \geq \delta$.*

Proof. By [17] $\dim \pi_G(S_r(X)) = r\dim(X)$, so π_G generically has $(r-1)$-dimensional fibers, which correspond to the choice of a point in $\mathbb{P}V$. This implies that $\pi_G|_{\pi_\sigma{}^{-1}(y)}$ is finite to one. Since $\dim \pi_\sigma^{-1}(y) \geq \delta$ we conclude. □

Corollary 3.2. *A border rank five tensor $T \in \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ has $\dim \underline{VSP}_{A\otimes B\otimes C}(T) \geq 8$.*

Proof. $\dim \sigma_5(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)) = 26$ [50]. □

Remark 3.3. In this case, by [54] T also has rank five and thus $\dim VSP_{A\otimes B\otimes C}(T) \geq 8$, where $VSP_{A\otimes B\otimes C}(T)$ is the variety of rank decompositions.

A similar argument shows:

Proposition 3.4. *Let $\mathcal{O}_{s,t,u}$ be a smallest dimensional G_T-orbit in $\mathbb{P}(S^sA\otimes S^tB\otimes S^uC)$. Then for all (s,t,u), $\dim \underline{VSP}(T) \geq \dim \mathcal{O}_{s,t,u}$.*

3.2. **How to find good tensors for the laser method?** The utility of a tensor $T \in A\otimes B\otimes C$ for the laser method may be thought of as the ratio of its *cost*, which is the asymptotic rank, $\underset{\sim}{\mathbf{R}}(T) := \lim_{N\to\infty}[\underline{\mathbf{R}}(T^{\boxtimes N})]^{1/N}$, and its *value*, which is its asymptotic subrank $\underset{\sim}{\mathbf{Q}}(T) := \lim_{N\to\infty}[\underline{\mathbf{Q}}(T^{\boxtimes N})]^{1/N}$. See [16] for a discussion, where the ratio of their logs is called the *irreversibility* of T. Here $\underline{\mathbf{Q}}(T)$ is the maximum q such that $M_{\langle 1\rangle}^{\oplus q} \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$, where $M_{\langle 1\rangle}^{\oplus q}$ is the so-called unit tensor, in bases $M_{\langle 1\rangle}^{\oplus q} = \sum_{j=1}^{q} a_j \otimes b_j \otimes c_j$. Unless a tensor is of minimal border rank, we only can estimate the asymptotic rank of a tensor by computing its border rank and the border rank of its small Kronecker powers.

There are several papers attempting to find tensors that give good upper bounds on ω in the laser method:

Papers on *barriers* may be interpreted as describing where *not* to look for good tensors: [16, 2, 4, 1] discuss limits of the laser method for various types of tensors and various types of implementations.

A program to utilize algebraic geometry and representation theory to find good tensors for the laser method was initiated in [19, 33].

Here we describe a more modest goal: determine criteria that indicate (or even guarantee) that border rank is strictly sub-multiplicative under the Kronecker square.

To our knowledge, the first example of a non-minimal border rank tensor that satisfied $\underline{\mathbf{R}}(T^{\boxtimes 2}) = \underline{\mathbf{R}}(T)^2$ was given in [18]: the small Coppersmith-Winograd tensor $T_{cw,q}$ for $q > 2$ and in this paper we show equality also holds when $q = 2$. This shows that tight tensors need not exhibit strict submultiplicativity. Several examples of strict submultiplicativity were known previous to this paper: the 2×2 matrix multiplication tensor $M_{\langle 2\rangle} \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, $\underline{\mathbf{R}}(M_{\langle 2\rangle}) = 7$ [29] while $\underline{\mathbf{R}}(M_{\langle 2\rangle}^{\boxtimes 2}) \leq 46$ [48]. The tensors of [15] have a drop of one, a generic tensor $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ satisfies $\underline{\mathbf{R}}(T) = 5$ while $\underline{\mathbf{R}}(T^{\boxtimes 2}) \leq 22$ [18], and $\underline{\mathbf{R}}(T_{skewcw,2}) = 5$ while $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 2}) = 17$ [18, 20].

3.3. $\underline{VSP}$ **and strict submultiplicativity.** All the strict submultiplicativity examples have positive dimensional $\underline{VSP}$. This is attributable to the degeneracy of $\sigma_4(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ for the generic tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, and to the large symmetry groups for the other cases: If a tensor $T \in A \otimes B \otimes C$ has a positive dimensional symmetry group G_T and G_T does not have a one-dimensional submodule in each of $A \otimes B$, $A \otimes C$, $B \otimes C$, $A \otimes B \otimes C$, then $\dim(\underline{VSP}(T)) > 0$ because any ideal in the G_T-orbit closure of an ideal of a border rank decomposition for T will give another border rank decomposition.

It would be too much to hope that a concise tensor T not of minimal border rank satisfying $\dim \underline{VSP}(T) > 0$ also satisfies $\underline{\mathbf{R}}(T^{\boxtimes 2}) < \underline{\mathbf{R}}(T)^2$. Consider the following example: Let $T = T_1 \oplus T_2$ with the T_j in disjoint spaces, where T_1 has non-minimal border rank and $\dim \underline{VSP}(T_1) = 0$ and T_2 has minimal border rank with $\dim \underline{VSP}(T_2) > 0$. Then there is no reason to believe $T^{\boxtimes 2}$ should have strict submultiplicativity.

It is possible that the converse holds: that strict submultiplicativity under the Kronecker square implies a positive dimensional $\underline{VSP}$.

It might be useful, following [15] to split the submultiplicativity question into two questions: first to determine if the usual tensor square is submultiplicative and then if the border rank of the Kronecker square is less than the border rank of the tensor square. Note that in general, assuming non-defectivity, for a projective variety $X \subset \mathbb{P}V$ of dimension N, $\sigma_{R-1}(X)$ has codimension $N+1$ in $\sigma_R(X)$. In our case $R = r^2$ and in the tensor square case $N = 6m-6$, and in the Kronecker square case $N = 3m^2 - 3$. A priori, for $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank r, $T^{\otimes 2} \in \sigma_{r^2}(Seg(\mathbb{P}^{(m-1)\times 6}))$ and submultiplicativity is a codimension $6m-5$ condition, whereas $T^{\boxtimes 2} \in \sigma_{r^2}(Seg(\mathbb{P}^{(m^2-1)\times 3}))$ and submultiplicativity is a codimension $3m^2-2$ condition. Despite this, the second condition is weaker than the first.

4. Koszul flattening lower bounds

The best general technique available for border rank lower bounds are Koszul flattenings [37, 35].

Fix an integer p. Given a tensor $T = \sum_{ijk} T^{ijk} a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C$, the p-th *Koszul flattening* of T on the space A is the linear map

$$T_A^{\wedge p} : \Lambda^p A \otimes B^* \to \Lambda^{p+1} A \otimes C$$
$$X \otimes \beta \mapsto \sum_{ijk} T^{ijk} \beta(b_j)(a_i \wedge X) \otimes c_k.$$

Then [35, Proposition 4.1.1] states

$$\underline{\mathbf{R}}(T) \geq \frac{\mathrm{rank}(T_A^{\wedge p})}{\binom{\dim(A)-1}{p}}. \tag{6}$$

The best lower bounds for any given p are obtained by restricting T to a generic $2p+1$ dimensional subspace of A^* so the denominator becomes $\binom{2p}{p}$.

Theorem 4.1. *The following border rank lower bounds are obtained by applying Koszul flattenings to a restriction of the tensor to a sufficiently generic $\mathbb{C}^{2p+1} \otimes B \otimes C \subset A \otimes B \otimes C$. Values of p that give the bound are in parentheses.*

(1) $\underline{\mathbf{R}}(T_{skewcw,4}^{\boxtimes 2}) \geq 39$ $(p = 2, 3, 4)$

(2) $\underline{\mathbf{R}}(T_{skewcw,6}^{\boxtimes 2}) \geq 70$ $(p = 2, 3, 4)$

(3) $\underline{\mathbf{R}}(T_{skewcw,8}^{\boxtimes 2}) \geq 110$ $(p = 4)$

(4) $\underline{\mathbf{R}}(T_{skewcw,10}^{\boxtimes 2}) \geq 157$ $(p = 4)$

(5) $\underline{\mathbf{R}}(T_{skewcw,2}^{\boxtimes 3}) \geq 49$ $(p = 4)$

(6) $\underline{\mathbf{R}}(T_{skewcw,4}^{\boxtimes 3}) \geq 219$ $(p = 3)$

(7) $\underline{\mathbf{R}}(T_{skewcw,6}^{\boxtimes 3}) \geq 550$ $(p = 3)$

(8) $\underline{\mathbf{R}}(T_{skewcw,8}^{\boxtimes 3}) \geq 1089$ $(p = 3)$

(9) $\underline{\mathbf{R}}(T_{skewcw,10}^{\boxtimes 3}) \geq 1886$ $(p = 3)$.

Better lower bounds for the larger cases are potentially possible, if not easily accessible, using larger values of p.

Compare these with the values for the small Coppersmith-Winograd tensor from [18]:

(1) $\underline{\mathbf{R}}(T_{cw,4}^{\boxtimes 2}) = 36$

(2) $\underline{\mathbf{R}}(T_{cw,6}^{\boxtimes 2}) = 64$

(3) $\underline{\mathbf{R}}(T_{cw,8}^{\boxtimes 2}) = 100$

(4) $\underline{\mathbf{R}}(T_{cw,10}^{\boxtimes 2}) = 144$

(5) $\underline{\mathbf{R}}(T_{cw,4}^{\boxtimes 3}) \geq 180$

(6) $\underline{\mathbf{R}}(T_{cw,6}^{\boxtimes 3}) = 512$

(7) $\underline{\mathbf{R}}(T_{cw,8}^{\boxtimes 3}) = 1000$

(8) $\underline{\mathbf{R}}(T_{cw,10}^{\boxtimes 3}) = 1728$.

Note that $\underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 4}) \leq (q+2)^4$ and that $\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes 4})$ is at least the estimate in Proposition 4.1 times $q+1$ by [18, Prop. 4.2]. Based on this, it is possible as of this writing that of $\underline{\mathbf{R}}(T_{skewcw,q}^{\boxtimes 4}) \leq \underline{\mathbf{R}}(T_{cw,q}^{\boxtimes 4})$ for $q = 2, 6, 8$.

5. Proof of Theorem 1.1 that $\underline{\mathbf{R}}(\mathrm{perm}_3) = 16$

The upper bound follows as $\underline{\mathbf{R}}(T_{cw,2}) = 4$.

For the lower bound, we prove there is no $E_{110} \subset A \otimes B$ of dimension 15 that satisfies the flag condition and passes the (210) and (120) tests. To do this, we need to show there is no E'_{110} of dimension six that is spanned by weight vectors such that the resulting E_{110} passes the tests and satisfies the flag condition. We do this by separately analyzing the pure and mixed kernels discussed in §2.5:

Lemma 5.1. *There is a unique up to isomorphism choice of six-dimensional $\mathbb{B}_{T_{cw,2}}$-fixed E'_{110} satisfying the flag condition with $\kappa_m = 5$ and this choice fails the (210)-test. All other choices satisfy $\kappa_m, \kappa'_m \leq 4$.*

Lemma 5.2. *For any six-dimensional E'_{110}, $\kappa_p, \kappa'_p \leq 10$ and if equality holds, then $\kappa_m, \kappa'_m < 4$.*

Lemmas 5.1 and 5.2 together prove Theorem 1.1.

In order to prove Lemma 5.1 we need to analyze what the flag condition imposes on potential E'_{110}. It turns out we only need to analyze the first three steps carefully. The essential point is when we analyze potential contributions to the mixed kernel, any element of the kernel carries a "cost" of at least three, in the sense that either three rank one weight vectors are used to construct the element of the kernel, or a weight vector that cannot appear until the third step in a flag is used, or a similar intermediate combination results in a cost of three.

In what follows, $\{i,j,k\} = \{1,2,3\}$, $\{i',j',k'\} = \{1,2,3\}$, $\widehat{k'} \in \{i',j'\}$, $\widehat{k} \in \{i,j\}$ etc..

Here

$$\mathrm{perm}_3(C^*) = \langle a^i_{i'} \otimes b^{\widehat{i}}_{\widehat{i'}} + a^{\widehat{i}}_{i'} \otimes b^i_{\widehat{i'}} + a^i_{\widehat{i'}} \otimes b^{\widehat{i}}_{i'} + a^{\widehat{i}}_{\widehat{i'}} \otimes b^i_{i'} \rangle.$$

Thus $\mathrm{perm}_3(C^*) \cap \sigma_3 = \emptyset$. Observe that $\kappa_f = 1$ as

$$(\mathrm{perm}_3(C^*) \otimes A) \cap (S^2 A \otimes B) = \langle \sum a^k_{k'} \otimes (a^i_{i'} \otimes b^j_{j'} + a^j_{i'} \otimes b^i_{j'} + a^i_{j'} \otimes b^j_{i'} + a^j_{j'} \otimes b^i_{i'}) \rangle.$$

Remark 5.3. In general, for any symmetric tensor T, $\kappa_f \geq 1$ due to the copy of T in $S^3 A \subset S^2 A \otimes B$.

The possible weights of elements in $A \otimes B$ are (200)(200), (110)(110), (200)(110) and their permutations under the action of $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}_2$. We will say an element has *type* $(xyz)(pqr)$ if its weight is in the $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}_2$-orbit of $(xyz)(pqr)$.

First step in flag: rank one weight vectors. All weight vectors of type (200)(200) have rank one, these are of the form $a^i_{i'} \otimes b^i_{i'}$. Vectors of type (200)(110) have rank one or two, those of rank one are of the form $a^i_{i'} \otimes b^i_{k'}$ and vectors of type (110)(110) have rank at most four, the rank one vectors among them are of the form $a^i_{i'} \otimes b^j_{j'}$.

Second step in the flag. Given the first step, we could get the second step either by adding another rank one weight vector, or taking a tangent vector to a rank one weight vector.

The rank two weight vectors tangent to a rank one element of type (200)(200), which we may write as $a^j_{j'}\otimes b^j_{j'}$, are up to scale $a^j_{j'}\otimes b^j_{\hat{j}'} + Ka^j_{\hat{j}'}\otimes b^j_{j'}$, for some $K \neq 0$, or its $\mathbb{Z}_2$-image, which are of type (200)(110) or $a^j_{j'}\otimes b^{\hat{j}}_{\hat{j}'} + Ka^{\hat{j}}_{\hat{j}'}\otimes b^j_{j'}$, which are of type (110)(110).

No rank two tangent vector to a rank one element of type (110)(110) is a weight vector.

The rank two tangent weight vectors to a rank one element of type (200)(110), e.g., $a^i_{i'}\otimes b^i_{k'}$, are of the form $a^i_{i'}\otimes b^{\hat{i}}_{k'} + Ka^{\hat{i}}_{i'}\otimes b^i_{k'}$ for some $K \neq 0$, and they are of type (110)(110).

Third step in the flag. Given a two-step flag, the third step may be obtained either by adding another rank one weight vector, or adding a new tangent vector to one of the rank one vectors in the flag, or taking a rank three second derivative of a weight vector, whose first tangent vector also appears in the flag, or by taking a vector of the form $x' + y'$ where x, y are rank one, appear in the flag, and are colinear.

Consider rank three second derivatives of a weight vector: Say we had some

$$(a + ta' + t^2a'')\otimes(b + tb' + t^2b'') = a\otimes b + t(a\otimes b' + a'\otimes b) + t^2(a\otimes b'' + a'\otimes b' + a''\otimes b) + ... \tag{7}$$

In order that the t coefficient appears in E'_{110}, either we would need either $a'\otimes b + a\otimes b'$ to be a weight vector, or one of $a\otimes b'$, $a'\otimes b$ to be a multiple of $a\otimes b$. The second case yields nothing interesting. To see this, assume $b' = \lambda b$. Then the coefficient of t^2 may be written as $a\otimes b'' + (\lambda a' + a'')\otimes b$, which is just a tangent vector to the original point, so we are just in the situation of a rank one point and two tangent vectors, so we ignore this situation.

Recalling the possible tangent vectors that are weight vectors we have three potential cases:

$$\langle a^j_{j'}\otimes b^j_{j'}, a^j_{j'}\otimes b^j_{\hat{j}'} + Ka^j_{\hat{j}'}\otimes b^j_{j'}, a^j_{j'}\otimes b'' + a^j_{\hat{j}'}\otimes b^j_{\hat{j}'} + Ka''\otimes b^j_{j'}\rangle$$
$$\langle a^j_{j'}\otimes b^j_{j'}, a^j_{j'}\otimes b^{\hat{j}}_{\hat{j}'} + Ka^{\hat{j}}_{\hat{j}'}\otimes b^j_{j'}, a''\otimes b^{\hat{j}}_{\hat{j}'} + a^{\hat{j}}_{\hat{j}'}\otimes b^{\hat{j}}_{\hat{j}'} + a^{\hat{j}}_{\hat{j}'}\otimes b''\rangle$$
$$\langle a^i_{i'}\otimes b^i_{k'}, a^i_{i'}\otimes b^{\hat{i}}_{k'} + Ka^{\hat{i}}_{i'}\otimes b^i_{k'}, a''\otimes b^{\hat{i}}_{k'} + a^{\hat{i}}_{i'}\otimes b^{\hat{i}}_{k'} + Ka^{\hat{i}}_{i'}\otimes b''\rangle$$

In the first case, the third vector is not a weight vector for any choice of a'', b'' not both zero, and in the other two cases the third vector can only be a weight vector if the third term collapses to a rank one element, and thus this scenario does not occur.

Since no two vectors of type (200)(200) are colinear, the only possibility of a vector of the form $x' + y'$ appearing is if x is of type (200)(200) and y of type (200)(110), but in this case one just gets a weight vector of the form x' of type (110)(110) by the weight vector requirement. If x, y are of type (200)(110), then any nontrivial $x' + y'$ is not a weight vector. Thus this scenario does not occur.

Proof of Lemma 5.1. We first describe all possible elements in the mixed kernel, these are given in equations (8) - (18). We then show that with a 6-dimensional E'_{110} satisfying the flag condition, no matter what combinations appear, there is at most a four dimensional contribution to the mixed kernel, with a unique (up to symmetry) exception whose pure kernel is too small for it to pass the (210) test.

Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Gamma \cdot a^j_{j'} \otimes b^k_{k'} = a^j_{j'} \otimes b^k_{k'} + a^j_{k'} \otimes b^k_{j'} + a^k_{j'} \otimes b^j_{k'} + a^k_{k'} \otimes b^j_{j'}$. In what follows underlined terms are elements of E'_{110}. The group G_{perm_3} allows us to unambiguously define the elements of E'_{110} except those of type (110)(110).

Up to $(\mathfrak{S}_3 \times \mathfrak{S}_3) \rtimes \mathbb{Z}_2$, there are four types of potential vectors in the mixed kernel that use a single element $\Gamma \cdot a^i_{i'} \otimes b^j_{j'}$ of $\mathrm{perm}_3(C^*)$:

First, E'_{110} could contain a complement to $\Gamma \cdot a^i_{i'} \otimes b^j_{j'}$ in its weight space (the complement is three dimensional). We have the following types:

$$a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'}) - a^j_{j'} \otimes \underline{(a^i_{i'} \otimes b^j_{j'})} - a^j_{j'} \otimes \underline{(a^i_{j'} \otimes b^j_{i'})} - a^j_{j'} \otimes \underline{(a^j_{i'} \otimes b^i_{j'})}, \tag{8}$$

$$a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'}) - a^j_{j'} \otimes \underline{(a^i_{i'} \otimes b^j_{j'} + a^i_{j'} \otimes b^j_{i'})} - a^j_{j'} \otimes \underline{(a^j_{i'} \otimes b^i_{j'})}, \tag{9}$$

$$a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'}) - a^j_{j'} \otimes \underline{(a^j_{i'} \otimes b^i_{j'} + a^i_{j'} \otimes b^j_{i'})} - a^j_{j'} \otimes \underline{(a^i_{i'} \otimes b^j_{j'})}, \tag{10}$$

$$a^j_{j'} \otimes (\Gamma \cdot a^i_{i'} \otimes b^j_{j'}) - a^j_{j'} \otimes \underline{(a^i_{i'} \otimes b^j_{j'} + a^i_{j'} \otimes b^j_{i'} + a^j_{i'} \otimes b^i_{j'})}. \tag{11}$$

We distinguish (9) and (10) because in (10) the rank two element appearing is tangent to a rank one weight vector of type (200)(200) and this is not the case in (9).

Note that the flag condition is crucial here: otherwise we could just take six vectors of the type appearing in (11) to have $\kappa_m = 6$.

The other potential elements of the mixed kernel utilizing a single element of $\mathrm{perm}_3(C^*)$ are (here and below, K, K' are constants):

$$a^j_{j'} \otimes (\Gamma \cdot a^{\hat{j}}_{\hat{j}'} \otimes b^j_{j'}) + a^{\hat{j}}_{\hat{j}'} \otimes \underline{(a^j_{j'} \otimes b^j_{j'})} + a^{\hat{j}}_{j'} \otimes \underline{(a^j_{j'} \otimes b^j_{\hat{j}'} + K a^j_{\hat{j}'} \otimes b^j_{j'})} + a^j_{\hat{j}'} \otimes \underline{(a^j_{j'} \otimes b^{\hat{j}}_{j'} + K a^{\hat{j}}_{j'} \otimes b^j_{j'})}, \tag{12}$$

where the underlined terms are respectively types (200)(200), (200)(110), (200)(110),

$$\begin{aligned} &a^i_{i'} \otimes (\Gamma \cdot a^k_{k'} \otimes b^j_{j'}) + a^k_{k'} \otimes \underline{(a^i_{i'} \otimes b^j_{j'} + K a^j_{j'} \otimes b^i_{i'})} + a^k_{j'} \otimes \underline{(a^i_{i'} \otimes b^j_{k'} + K' a^j_{k'} \otimes b^i_{i'})} \\ &\quad + a^j_{k'} \otimes \underline{(a^i_{i'} \otimes b^k_{j'} + K' a^k_{j'} \otimes b^i_{i'})} + a^j_{j'} \otimes \underline{(a^i_{i'} \otimes b^k_{k'} + K a^k_{k'} \otimes b^i_{i'})}, \end{aligned} \tag{13}$$

where the underlined terms are all of type (110)(110) and they all live in different weight spaces, and

$$\begin{aligned} &a^j_{i'} \otimes (\Gamma \cdot a^{\hat{j}}_{k'} \otimes b^j_{j'}) + a^{\hat{j}}_{k'} \otimes \underline{(a^j_{i'} \otimes b^j_{j'} + K a^j_{j'} \otimes b^j_{i'})} + a^{\hat{j}}_{j'} \otimes \underline{(a^j_{i'} \otimes b^j_{k'} + K' a^j_{k'} \otimes b^j_{i'})} \\ &\quad + a^j_{k'} \otimes \underline{(a^j_{i'} \otimes b^{\hat{j}}_{j'} + K' a^{\hat{j}}_{j'} \otimes b^j_{i'})} + a^j_{j'} \otimes \underline{(a^j_{i'} \otimes b^{\hat{j}}_{k'} + K a^{\hat{j}}_{k'} \otimes b^j_{i'})}, \end{aligned} \tag{14}$$

where the underlined terms are respectively of types (200)(110), (200)(110), (110)(110), (110)(110). The two elements of type (110)(110) live in different weight spaces.

The possible terms in the mixed kernel with two elements of $\mathrm{perm}_3(C^*)$ that are not just sums of terms with one element having no cancellation are as follows:

$$
\begin{aligned}
&a^i_i\otimes(\Gamma\cdot a^k_k\otimes b^i_j)+a^i_k\otimes(\Gamma\cdot a^i_i\otimes b^k_j)\\
&+a^k_k\otimes\underline{(a^i_i\otimes b^i_j+Ka^i_j\otimes b^i_i)}+a^k_j\otimes\underline{(a^i_i\otimes b^i_k+a^i_k\otimes b^i_i)}+a^k_i\otimes\underline{(a^i_k\otimes b^i_j)}+a^i_j\otimes\underline{(a^i_i\otimes b^k_k+Ka^k_k\otimes b^i_i)}+a^i_j\otimes\underline{(a^i_k\otimes b^k_i)},
\end{aligned}
\tag{15}
$$

where the underlined terms are respectively of types (200)(110), (200)(110), (200)(110), (110)(110), (110)(110) and the two type (110)(110) vectors live in different weight spaces, and

$$
\begin{aligned}
&a^i_i\otimes(\Gamma\cdot a^k_k\otimes b^i_j)+a^i_j\otimes(\Gamma\cdot a^i_i\otimes b^k_k)\\
&+a^k_k\otimes\underline{(a^i_i\otimes b^i_j+a^i_j\otimes b^i_i)}+a^k_j\otimes\underline{(a^i_i\otimes b^i_k+K'a^i_k\otimes b^i_i)}+a^i_k\otimes\underline{(a^i_i\otimes b^k_j+K'a^k_j\otimes b^i_i)}+a^i_k\otimes\underline{(a^i_j\otimes b^k_i)}+a^k_i\otimes\underline{(a^i_j\otimes b^i_k)},
\end{aligned}
\tag{16}
$$

where the underlined terms are respectively of types (200)(110), (200)(110), (110)(110), (110)(110) and the two type (110)(110) vectors live in different weight spaces.

The possible terms in the mixed kernel with three elements of $\mathrm{perm}_3(C^*)$ are:

$$
\begin{aligned}
&a^i_{i'}\otimes(\Gamma\cdot a^{\hat{i}}_{k'}\otimes b^i_{j'})+a^i_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^{\hat{i}}_{j'})+a^i_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^{\hat{i}}_{k'})\\
&+a^{\hat{i}}_{k'}\otimes\underline{(a^i_{i'}\otimes b^i_{j'}+a^i_{j'}\otimes b^i_{i'})}+a^{\hat{i}}_{j'}\otimes\underline{(a^i_{i'}\otimes b^i_{k'}+a^i_{k'}\otimes b^i_{i'})}+a^{\hat{i}}_{i'}\otimes\underline{(a^i_{k'}\otimes b^i_{j'}+a^i_{j'}\otimes b^i_{k'})},
\end{aligned}
\tag{17}
$$

where all have type (200)(110), with the same (200) weight space and the three distinct (110) weight spaces.

$$
\begin{aligned}
&a^i_{i'}\otimes(\Gamma\cdot a^j_{j'}\otimes b^k_{k'})+a^j_{j'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^k_{k'})+a^k_{k'}\otimes(\Gamma\cdot a^i_{i'}\otimes b^j_{j'})\\
&+a^i_{i'}\otimes\underline{(a^j_{k'}\otimes b^k_{j'}+a^k_{j'}\otimes b^j_{k'})}+a^j_{j'}\otimes\underline{(a^i_{k'}\otimes b^k_{i'}+a^k_{i'}\otimes b^i_{i'})}+a^k_{k'}\otimes\underline{(a^i_{j'}\otimes b^j_{i'}+a^j_{i'}\otimes b^i_{j'})},
\end{aligned}
\tag{18}
$$

where all have type (110)(110) and lie in different weight spaces.

We first observe that no element appears in more than 4 of the potential kernel vectors. Since we only have six elements to obtain a 5 dimensional mixed kernel, we see any potential kernel vector using more than 3 elements of E'_{110} cannot be used. (Here we are also using that with vectors involving more than 3 elements, no pair of elements appears in a same second potential kernel vector.) We are reduced to examining (8), (9), (11), (12), (17) and (18).

Case: the kernel contains an element of type (8). We have three elements of type (110)(110) all of the same weight in E'_{110}. The only other relations among those we consider that use elements of type (110)(110) are of type (18), but this requires two additional elements of that type with different weights. We conclude elements of type (8) will not be useful.

Case: the kernel contains an element of type (9) or (10). The second underlined term is not tangent to the first, so the cost of such an element is three. If the element is obtained naively, we are in the same situtation as (8). For case (10) there is also the possibility to obtain the rank two element as a tangent vector to an element of type (200)(200). If we form the 3-plane with the (200)(200) vector and want to use it in some other relation, we cannot reuse the (110)(110) vectors with it as no other kernel element involves this mixture of weights, so these cases are not useful.

Case: the kernel contains an element of type (11). We have seen there is no way to construct the element appearing at cost three other than as the sum of three rank one elements or as the sum of a rank one element and a tangent vector as above. Thus this case reduces to one of the previous two cases.

Case: the kernel contains an element of type (12). These use an element of type (200)(200) and two of its type (200)(110) tangent vectors. Each (200)(200) element appears in 4 relations of type (12). To obtain them, we only need include the four vectors $a^j_{j'}\otimes b^j_{\widehat{j'}} + a^j_{\widehat{j'}}\otimes b^j_{j'}$, $a^j_{j'}\otimes b^{\widehat{j}}_{j'} + K a^{\widehat{j}}_{j'}\otimes b^j_{j'}$ (where there are two possibilities for each of the the two hatted vectors). When $K = 1$, this gives $\kappa_m, \kappa'_m = 4$ (if $K \neq 1$, then κ'_m is smaller). At this point we have a five-plane in E'_{110}. We could take a (200)(110)-vector appearing as a tangent vector to the (200)(200) vector and attempt to use it in an element of type (17), in fact we may use two tangent vectors in such an expression, but we still need to add a third (200)(110) vector that is not of rank one, and is not tangent to the (200)(200) vector. Explicitly, we may take

$$E'_{110} = \langle a^1_1\otimes b^1_1, a^1_1\otimes b^1_2 + a^1_2\otimes b^1_1, a^1_1\otimes b^1_3 + a^1_3\otimes b^1_1, a^1_1\otimes b^2_1 + a^2_1\otimes b^1_1, a^1_1\otimes b^3_1 + a^3_1\otimes b^1_1, a^1_2\otimes b^1_3 + a^1_3\otimes b^1_2\rangle \tag{19}$$

Here the (210) test fails, $\kappa_p = 6$ (for each basis vector v of E'_{110} there is a pure kernel of the form $a^1_1\otimes v + \cdots$), so the total kernel has dimension 12.

Case: the kernel contains an element of type (17). Each element is a tangent vector to a same vector of type (200)(200), namely $a^i_{i'}\otimes b^i_{i'}$. Thus to get just one element of the kernel already uses 4 dimensions of E'_{110}. Unless we enlarge to obtain (19), we would have to enlarge to obtain either additional relations of type (17) or relations of type (18). To get a second such relation, one needs at least two more elements, which would fill E'_{110} and only give $\kappa_m = 3$. The terms in (18) have a different type, so that will not work either.

Case: the kernel contains an element of type (18). The elements appearing are all of type (110)(110) and in different weight spaces so at most one of these could be re-used in a different relation, so this situation cannot occur. □

For the proof of Lemma 5.2 we use a basic fact from exterior differential systems (the easy part of Cartan's test) [28, Prop. 4.5.3]: let $B_1 \subset B_2 \subset \cdots \subset B_9$ be a generic flag in B (generic in the sense that s_1 below is maximized, and having maximized s_1, s_2 is maximized etc..). Let s_1 be the dimension of the projection of E'_{110} to $A\otimes B_1$. Define s_2 by $s_1 + s_2$ is dimension of the projection of E'_{110} to $A\otimes B_2$, set $s_1 + s_2 + s_3$ to be the dimension of E'_{110} projected to $A\otimes B_3$ etc.. Then

$$\dim(S^2A\otimes B) \cap (A\otimes E'_{110}) \leq s_1 + 2s_2 + \cdots + 6s_6. \tag{20}$$

In particular, $\kappa_p \leq s_1 + 2s_2 + \cdots + 6s_6$. If equality holds in (20), we will say E'_{110} is A-involutive.

Proof of Lemma 5.2. Let $t_1, \cdots, t_6$ be the corresponding quantities for the (120) test. The only way to have $\kappa_p, \kappa'_p \geq 10$ is if the associated Young diagrams are respectively A and B involutive, and both the staircase, i.e., $s_1 = t_1 = 3, s_2 = t_2 = 2, s_3 = t_3 = 1$. This is because involutivity can only hold if E'_{110} is spanned by rank one elements and in this case the B-diagram is the transpose of the A-diagram.

Thus there are $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$ such that

$$E'_{110} = \langle a_1\otimes b_1, a_1\otimes b_2, a_2\otimes b_1, a_1\otimes b_3, a_3\otimes b_1, a_2\otimes b_2\rangle.$$

The best one can do here is to obtain $\kappa_m = 1$, e.g., by taking $a_1 = a_1^1$, $a_2 = a_3^1$, $a_3 = a_1^3$ and similarly for the b_j. □

Remark 5.4. The reason perm_3 was previously unaccessible was that already to choose E'_{110}, without the flag condition one needed to introduce numerous parameters due to the high weight multiplicities that made the calculation infeasible. The flag condition guaranteed the presence of low rank elements in E'_{110} which significantly reduced the search space.

Remark 5.5. It is interesting to see what happens when $\dim E'_{110} = 7$, to obtain a border rank 16 ideal fixed by the torus in G_{perm_3}. One may take for example

$$E'_{110} = \langle a_1^1 \otimes b_1^1, a_2^1 \otimes b_1^1 + a_1^1 \otimes b_2^1, a_3^1 \otimes b_1^1 + a_1^1 \otimes b_3^1, a_1^2 \otimes b_1^1 + a_1^1 \otimes b_1^2, a_1^3 \otimes b_1^1 + a_1^1 \otimes b_1^3, a_2^1 \otimes b_2^1, a_3^1 \otimes b_2^1 + a_2^1 \otimes b_3^1 \rangle.$$

Then we obtain the four (200)(200) contributions to κ_m from expressions of type (12) as well as three additional contributions from expressions of type (17). Here $s_1 = t_1 = 4$, $s_2 = t_2 = 3$ and

$$\begin{aligned}
&(A \otimes E'_{110}) \cap (S^2 A \otimes B) = \\
&\langle a_1^1 \otimes a_1^1 \otimes b_1^1, a_2^1 \otimes a_1^1 \otimes b_1^1 + a_1^1 \otimes (a_2^1 \otimes b_1^1 + a_1^1 \otimes b_2^1), a_3^1 \otimes a_1^1 \otimes b_1^1 + a_1^1 \otimes (a_3^1 \otimes b_1^1 + a_1^1 \otimes b_3^1), \\
&a_1^2 \otimes a_1^1 \otimes b_1^1 + a_1^1 \otimes (a_1^2 \otimes b_1^1 + a_1^1 \otimes b_1^2), a_1^3 \otimes a_1^1 \otimes b_1^1 + a_1^1 \otimes (a_1^3 \otimes b_1^1 + a_1^1 \otimes b_1^3), \\
&a_2^1 \otimes a_2^1 \otimes b_2^1, a_1^1 \otimes a_2^1 \otimes b_2^1 + a_2^1 \otimes (a_2^1 \otimes b_1^1 + a_1^1 \otimes b_2^1), a_3^1 \otimes a_2^1 \otimes b_2^1 + a_2^1 \otimes (a_3^1 \otimes b_2^1 + a_2^1 \otimes b_3^1) \rangle
\end{aligned}$$

so $\kappa_p = \kappa'_p = 8$ and both the (210) and (120) tests are passed.

6. Descriptions of $\underline{VSP}(T_{cw,q})$

In this section we adopt the index range $1 \le \alpha, \beta \le q$. The small Coppersmith-Winograd tensor has a well-known border rank decomposition, which is also a Waring border rank decomposition.

$$\begin{aligned}
T_{cw,q} = &\lim_{t \to 0} \\
&\frac{1}{t^2} \sum_\alpha [(a_0 + t a_\alpha) \otimes (b_0 + t b_\alpha) \otimes (c_0 + t c_\alpha)] \\
&- \frac{1}{t^3} \left[(a_0 + t^2 \sum_\alpha a_\alpha) \otimes (b_0 + t^2 \sum_\alpha b_\alpha) \otimes (c_0 + t^2 \sum_\alpha c_\alpha) \right] \\
&- (q \frac{1}{t^2} - \frac{1}{t^3}) a_0 \otimes b_0 \otimes c_0.
\end{aligned}$$

Let $q > 2$. Write $A = B = C = L \oplus M$, where $L = \langle a_0 \rangle$ and $M = \langle a_\alpha \rangle$. Set $Q = \sum_\alpha a_\alpha \otimes a_\alpha$. A straight-forward Lie algebra calculation (see, e.g., [19]) shows $G_{T_{cw,q}} \supset SO(M,Q) \times GL(L) = SO(q) \times \mathbb{C}^*$. Then

$$A \otimes B = L^{\otimes 2} \oplus L \wedge M \oplus S_0^2 M \oplus \Lambda^2 M \oplus (L \cdot M \oplus Q),$$

where the term in parenthesis is $T_{cw,q}(C^*)$. Here $S_0^2 M = M_{2\omega_1}$ is the complement to the trivial $SO(M,Q)$-representation in $S^2 M$. In what follows we write L^k for $L^{\otimes k} = S^k L$.

Theorem 6.1. *For $q > 2$, $\underline{VSP}(T_{cw,q})$ is a point. The unique ideal is as follows: for all s,t,u with $s+t+u = d$, the annhilator of the ideal in degree (s,t,u) is*

$$L^d \oplus L^{d-1} \cdot M \oplus L^{d-2} \cdot Q.$$

Here

$$L^{d-1} \cdot M = \langle a_0^{s-1} \cdot a_\alpha \otimes b_0^t \otimes c_0^u + a_0^s \otimes b_0^{t-1} \cdot b_\alpha \otimes c_0^u + a_0^s \otimes b_0^t \otimes c_0^{u-1} \cdot c_\alpha \mid \alpha = 1, \cdots, q \rangle$$

and

$$L^{d-2}\cdot Q=\langle \sum_\alpha a_0^{s-1}\cdot a_\alpha \otimes b_0^{t-1}\cdot b_\alpha\otimes c_0^u + a_0^{s-1}\cdot a_\alpha\otimes b_0^t\otimes c_0^{u-1}\cdot c_\alpha + a_0^s\otimes b_0^{t-1}\cdot b_\alpha\otimes c_0^{u-1}\cdot c_\alpha$$
$$+ a_0^{s-2}\cdot a_\alpha^2\otimes b_0^t\otimes c_0^u + a_0^s\otimes b_0^{t-2}\cdot b_\alpha^2\otimes c_0^{u-1}\cdot c_\alpha + a_0^s\otimes b_0^t\otimes c_0^{u-2}\cdot c_\alpha^2\rangle.$$

Proof. We must have $\mathbb{P}E_{110}\cap Seg(\mathbb{P}A\times\mathbb{P}B)\neq\emptyset$. This may be achieved by adding some

$$(u_0a_0+\sum_\alpha u_\alpha a_\alpha)\otimes(v_0b_0+\sum_\beta v_\beta b_\beta)$$

for $u_0,u_\alpha,v_0,v_\beta\in\mathbb{C}$. We also must have a flag as in Observation 2.5. Taking anything other than $a_0\otimes b_0$, $(u_0a_0+a)\otimes b_0$ with $u_0\in\mathbb{C}$, $a\in M$, or $xa_0\otimes b_\alpha+ya_\alpha\otimes b_0$ (i.e., some $a_0\otimes b_\alpha$ or $a_\alpha\otimes b_0$ since we are working modulo $T(C^*)$) makes the flag condition $\mathbb{P}F_2\subset\sigma_2(Seg(\mathbb{P}A\times\mathbb{P}B))$ fail. (Here we use that $q>2$.) Taking anything other than $a_0\otimes b_0$ makes the flag condition $\mathbb{P}F_3\subset\sigma_3(Seg(\mathbb{P}A\times\mathbb{P}B))$ fail. Thus there is a unique E_{110}, and by symmetry unique E_{101} and E_{011}. This triple exactly passes all degree three tests.

To see E_{200} must be as asserted, it must be such that $(E_{200}\otimes B)\supseteq E_{210}$. In order to have $L^{\otimes 3}$ in this intersection, we need $L^{\otimes 2}\subset E_{200}$. In order to have $L^2\cdot M=\langle a_0\otimes a_0\otimes b_\alpha+a_0\otimes a_\alpha\otimes b_0+a_\alpha\otimes a_0\otimes b_0\rangle$ in the intersection, we see it must also contain $\langle a_0\otimes a_\alpha+a_\alpha\otimes a_0\rangle=L\cdot M$. In order to have $L\cdot Q=\langle\sum_\alpha(a_0\otimes a_\alpha\otimes b_\alpha+a_\alpha\otimes a_0\otimes b_\alpha+a_\alpha\otimes a_\alpha\otimes b_0)\rangle$ in the intersection, we see it must also contain $\langle\sum_\alpha a_\alpha\otimes a_\alpha\rangle=Q$.

For the general case, assume by induction $E_{s-1,t,u},E_{s,t-1,u},E_{s,t,u-1}$ are as asserted and isomorphic as a module to $L^{\otimes d-1}\oplus L^{d-2}\cdot M\oplus L^{d-3}\cdot Q$. Arguing as we did for E_{200}, first obtaining $L^{\otimes d}$, then $L^{d-1}\cdot M$, then $L^{d-2}\cdot Q$ we conclude. □

Note that the ideal is $G_{T_{cw,q}}$-fixed as indeed it has to be if $\underline{VSP}$ is a point.

Now let $q=2$, in this case it is more convenient to write $T_{cw,2}$ as

$$T_{cw,2}=\sum_{\sigma\in\mathfrak{S}_3}a_{\sigma(1)}\otimes b_{\sigma(2)}\otimes c_{\sigma(3)}.$$

Write $A=B=C=L_1\oplus L_2\oplus L_3$ where, e.g., for A, $L_j=\langle a_j\rangle$. A straight-forward Lie algebra calculation shows $\widehat{G}_{T_{cw,2}}\supseteq(\mathbb{C}^*)^{\times 3}$.

Theorem 6.2. *$\underline{VSP}(T_{cw,2})$ and $\underline{VSP}_{v_3(\mathbb{P}^2),\mathbb{P}S^3\mathbb{C}^3}(T_{cw,2})$ each consists of three points. One choice has for all s,t,u with $s+t+u=d$, the annihilator in degree (s,t,u) equal to*

$$L_1^s\otimes L_1^t\otimes L_1^u\oplus\phi(L_1^{d-1}\otimes L_2)\oplus\phi(L_1^{d-1}\otimes L_3)\oplus\phi(L_1^{s-2}\otimes L_2\otimes L_3)$$

where $\phi:(L_1^{d-1}\otimes L_x)\to S^sA\otimes S^tB\otimes S^uC$ is the symmetric embedding. The other two choices arise from exchanging the role of L_1 with L_2,L_3.

Proof. We have $T_{cw,2}(C^*)=\langle a_i\otimes b_j+b_j\otimes a_i\mid i\neq j\rangle$. The only possibilites for E_{110} for $r=4$ that pass the (210)-test arise by adding $a_k\otimes b_k$ to this for some $k\in\{1,2,3\}$. Take $k=1$. Then

$$(E_{110}\otimes A)\cap(S^2A\otimes B)=\langle a_1^2\otimes b_1,a_1a_2\otimes b_1+a_1^2\otimes b_2,a_1a_3\otimes b_1+a_1^2\otimes b_3,\sum_{\sigma\in\mathfrak{S}_3}a_{\sigma(1)}\otimes a_{\sigma(2)}\otimes b_{\sigma(e)}\rangle$$

The only compatible choice of E_{200} is $\langle a_1^2,a_1a_2,a_1a_3,a_2a_3\rangle$. The situation for higher multi-degrees is similar. □

Remark 6.3. In contrast to $T_{cw,2}$, by Corollary 3.2, $\dim \underline{VSP}(T_{skewcw,2}) \geq 8$. From [23] (slightly changing notation) we have the rank five decomposition:

$$T_{skewcw,2} = \frac{1}{2}[2a_1 \otimes (b_2 - b_3) \otimes (c_2 + c_3) - (a_1 + a_2) \otimes (b_1 - b_3) \otimes (c_1 + c_3) - (a_1 - a_2) \otimes (b_1 + b_3) \otimes (c_1 - c_3) + (a_1 + a_3) \otimes (b_1 - b_2) \otimes (c_1 + c_2) - (a_1 - a_3) \otimes (b_1 + b_2) \otimes (c_1 - c_2)]$$

and the orbit of this decomposition already has dimension 8. (This can be seen by noting that more than four distinct vectors in $\mathbb{C}^3$ appear in the decomposition.)

7. $T_{skewcw,q}$, $q > 2$

Proof of Theorem 1.3. For the upper bound, we have

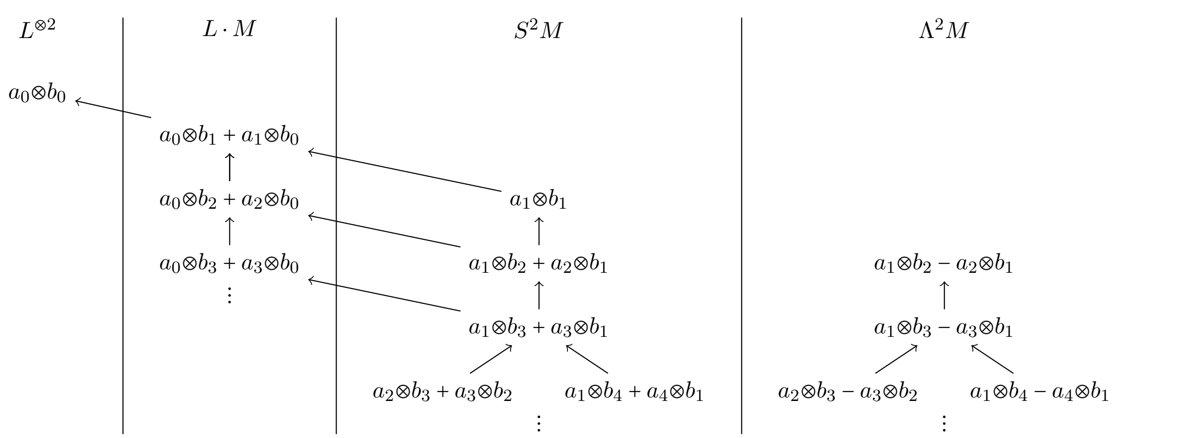
(21)
$$T_{skewcw,q} = \lim_{t \to 0} \frac{1}{t^3}[\sum_{\xi}[(a_0 + t^2 a_\xi) \otimes (b_0 - t^2 b_\xi) \otimes (c_0 - t c_{\xi+p}) + (a_0 - t^2 a_\xi) \otimes (b_0 - t b_{\xi+p}) \otimes (c_0 + t^2 c_\xi) + (a_0 - t a_{\xi+p}) \otimes (b_0 + t^2 b_\xi) \otimes (c_0 - t^2 c_\xi)] + \frac{1}{t^5}(a_0 + t^3 \sum_{\xi} a_{\xi+p}) \otimes (b_0 + t^3 \sum_{\xi} b_{\xi+p}) \otimes (c_0 + t^3 \sum_{\xi} c_{\xi+p}) - (\frac{3q}{2t^2} + \frac{1}{t^5}) a_0 \otimes b_0 \otimes c_0].$$

For the lower bounds, write $A = B = C = L \oplus M$ with $\dim L = 1$, $\dim M = q$ and M is equipped with a symplectic form Ω. A straight-forward Lie algebra calculation shows $G_{T_{skewcw,q}} \supseteq Sp(M) \times GL(L) \times M^* \otimes L$. Then

$$A \otimes B = L^{\otimes 2} \oplus L \cdot M \oplus S^2 M \oplus \Lambda^2 M_0 \oplus (L \wedge M \oplus \Omega)$$

where the term in parentheses equals $T_{skewcw,q}(C^*)$. Here $\Lambda^2 M_0 = M_{\omega_2}$, the complement to the $Sp(M)$-trivial representation in $\Lambda^2 M$.

we have the following weight diagram for the $G_{T_{skewcw,q}}$-complement of $T(C^*)$ in $A \otimes B$:



We will show that for $q \leq 10$, there is no choice of E'_{110} satisfying all degree three tests when $r = \frac{3}{2}q + 1$. We focus on the case $q = 10$ as that is the most difficult, the other cases are easier.

Note that elements of M may be raised to L, so an element of S^2M cannot be placed in E'_{110} unless its raising to $L\cdot M$ is also there. On the other hand, since $L\wedge M\subset E_{110}$, there is no similar restriction on elements of $\Lambda^2 M$.

We now restrict to $q=10$. We split the types of (110) spaces into 10 types of cases depending on the dimension of E'_{110} intersected with the various irreducible modules:

case	$L^{\otimes 2}$	$L\cdot M$	S^2M	Λ^2_0M
1	1	4	0	0
2	1	3	1	0
3	1	2	2	0
$4x$	1	2	$1+\frac{1}{2}$	$\frac{1}{2}$
5	0	0	0	5
6	1	0	0	4
7	1	1	0	3
8	1	2	0	2
9	1	1	1	2
10	1	2	1	1

Types $1,2,3,8,9,10$ are all single cases Types $5,6,7$ each involve a choice of subset of weight vectors in Λ^2_0M (so they are each a collection of a finite number of cases) and case 4 involves a parameter, where we use $\frac{1}{2}$ to indicate the parameter, as the weight vector is a sum of a vector in the two indicated spaces. Explicitly, case $4x$ may be written

$$E'_{110}=\langle a_0\otimes b_0,a_0\otimes b_1+a_1\otimes b_0,a_0\otimes b_2+a_2\otimes b_0,a_1\otimes b_1,x(a_1\otimes b_2+a_2\otimes b_1)+a_1\otimes b_2-a_2\otimes b\rangle.$$

Of these cases $1,2,3,4x,8,10$ pass the (210) and (120) tests. No triple passes the (111) test. □

We remark that the decomposition (21) is $\mathbb{Z}_3$-invariant.

Corollary 7.1. *For $10\geq q>2$, and $q=2p$ even, $\underline{VSP}(T_{skewcw,q})$ contains the isotropic Grassmannian $G_\Omega(\frac{q}{2},M)$. In particular it has dimension at least $\binom{p}{2}$.*

Proof. Examining (21), by $Sp(M)\subset G_{T_{skewcw,q}}$ we may replace $\langle a_\xi\rangle$ with any isotropic subspace as long as we replace $\langle a_{\xi+p}\rangle$ with the corresponding dual subspace and the same changes in B,C. □

8. A simpler Waring border rank 17 expression for $\det_3$

In this section and the next, we present explicit decompositions. The method used to obtain the decompositions is discussed after the second decomposition at the end of §9.

Set $i = \sqrt{-1}$ and $\zeta = e^{2\pi i/12}$. Then $\det_3 = \sum_{s=1}^{17} m_s^{\otimes 3}(t) + O(t)$, where the m_s are the following matrices

$$\begin{pmatrix} \frac{\zeta^6}{t^5} & 0 & 0 \\ 0 & \zeta^6 & 0 \\ 0 & 0 & t^5 \end{pmatrix} \begin{pmatrix} \frac{1}{t^5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^6}{t^5} & 0 & 0 \\ 0 & 0 & t\zeta^8 \\ 0 & t^4\zeta^4 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^4}{t^5} & 0 & 0 \\ 0 & 0 & t\zeta^6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^5}{t^5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t^4 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^3}{t^5} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^5 \end{pmatrix} \begin{pmatrix} 0 & \frac{\zeta^{10}}{t^4} & \frac{\zeta^8}{t^3} \\ 0 & 0 & t\zeta^8 \\ t^3\zeta^6 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\zeta^8}{t^4} & \frac{\zeta^6}{t^3} \\ 0 & 0 & t\zeta^6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{t^4} & \frac{1}{t^3} \\ 0 & 0 & 0 \\ t^3\zeta^6 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\zeta^6}{t^4} & 0 \\ \frac{\zeta^6}{t} & 0 & 0 \\ 0 & 0 & t^5\zeta^6 \end{pmatrix} \begin{pmatrix} 0 & \frac{\zeta^{11}}{t^4} & 0 \\ 0 & 0 & 0 \\ t^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{\zeta^9}{t^4} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^5\zeta^6 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \frac{1}{t^3} \\ \frac{\zeta^6}{t} & 0 & 0 \\ 0 & t^4\zeta^6 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{t^3} \\ 0 & \zeta^4 & 0 \\ t^3\zeta^2 & t^4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{\zeta^6}{t^3} \\ 0 & \zeta^{10} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{5^{\frac{5}{6}}\zeta^2}{5t^4} & \frac{\left(1+\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^2}{t^3} \\ \frac{\left(1-\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^8}{t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{5^{\frac{5}{6}}\zeta^8}{5t^4} & \frac{\left(1-\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^2}{t^3} \\ \frac{\left(1+\frac{2}{5}\sqrt{5}\right)^{\frac{1}{3}}\zeta^8}{t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

9. A numerical border rank 42 expression for $T_{skewcw,4}^{\boxtimes 2}$

What follows is an expression for $T_{\text{skewcw},4}^{\boxtimes 2}$ as $\sum_{s=1}^{42} m_s(t)^{\otimes 3} + O(t)$ that is satisfied to an error of at most 4.4×10^{-15} in each entry. Code to verify the assertion is available at https://www.math.tamu.edu/~jml/chllasersupp.html.

It consists of 42 matrices whose entries are rational expressions in the following 36 complex numbers: Let $i = \sqrt{-1}$ and let $\zeta = e^{2\pi i/12}$. Set

$$\begin{aligned}
z_0 &= -0.8660155098072051 + 0.9452855522785384i & z_1 &= -1.2981710770246242 + 0.0008968724089185688i \\
z_2 &= 2.9260271139931078 + 0.1853833642730014i & z_3 &= 0.2542517122150322 + 0.3079381943837828 4i \\
z_4 &= 0.6964375578992822 + 0.2772662627986198i & z_5 &= 0.5507020325318998 - 0.0493931308002328i \\
z_6 &= 1.149228383831849 - 1.1683147648642283i & z_7 &= 0.6586404058476252 - 0.16578044112199047i \\
z_8 &= 0.7654345273805864 - 0.06877274843008892i & z_9 &= 0.544690883860558 + 0.09720573163212605i \\
z_{10} &= 0.6932236636741451 + 0.14980159446358277i & z_{11} &= 0.5862637032385472 - 0.12844523449559558i \\
z_{12} &= 2.384363992555291 - 0.08927102369428247i & z_{13} &= 0.9664252976479286 + 0.08480470055107503i \\
z_{14} &= 0.6190926897383283 + 0.15631000400545272i & z_{15} &= 0.6283592253932955 - 0.5626050553495663i \\
z_{16} &= 1.8190778570602204 - 0.22163457440913656i & z_{17} &= 1.153187286528645 - 0.07977233251120702i \\
z_{18} &= 1.4498877801613976 - 0.22515738202335905i & z_{19} &= 0.7262464450114047 + 0.7050051641972112i \\
z_{20} &= 1.1195537528292199 - 0.26381000320340176i & z_{21} &= 0.4400325048210471 + 0.6593492930106759i \\
z_{22} &= 0.3476654993676339 + 0.4095417606798612i & z_{23} &= 0.9459769225333798 + 0.24589162882727128i \\
z_{24} &= 0.7637135867709066 - 0.10529269213820387i & z_{25} &= 0.7409392923310902 - 0.10474756303325146i \\
z_{26} &= 1.0112068238001992 - 0.12695675940574122i & z_{27} &= 1.5005677845016696 - 0.24533651960180036i \\
z_{28} &= 0.6134145054919202 + 0.08121891266185506i & z_{29} &= 1.145625294745251 - 0.3813562005184122i \\
z_{30} &= 1.0607612533915372 - 0.016294891090460426i & z_{31} &= 0.941339345482511 + 0.20413704882122435i \\
z_{32} &= 0.622575977639622 + 0.2555810563389569i & z_{33} &= 0.951746321194872 - 0.2894768358835511i \\
z_{34} &= 1.0532801812660977 - 0.2502246606675517i & z_{35} &= 1.0207644184200035 - 0.2106937666100475i.
\end{aligned}$$

The 42 matrices are:

$$\begin{pmatrix} \frac{\zeta^3 z_{25}z_{26}z_{31}z_{35}}{t^{269}} & \frac{\zeta^8 z_{28}t^{61}}{z_{23}z_{25}z_{26}z_{31}z_{33}z_{34}} & \frac{\zeta^{10}z_{24}z_{35}t^{13}}{z_{23}z_{25}z_{26}z_{31}z_{33}z_{34}} & 0 & 0 \\ \frac{z_{25}z_{26}z_{31}z_{33}z_{35}}{t^{104}} & 0 & 0 & 0 & \frac{\zeta^9 z_{18}z_{26}z_{32}t^{148}}{z_{21}z_{23}z_{33}z_{35}^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^7 z_{35}t^{121}}{z_{23}z_{25}z_{26}^2z_{31}z_{33}z_{34}} & 0 & \frac{\zeta^8 z_{23}z_{34}t^{91}}{z_{24}z_{35}^2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^{10}z_{30}}{t^{269}} & 0 & \frac{\zeta^{10}z_{24}z_{35}t^{13}}{z_{27}z_{28}z_{30}z_{33}} & 0 & 0 \\ 0 & 0 & \frac{\zeta^7 z_{24}z_{35}t^{178}}{z_{27}z_{28}z_{30}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^6 z_{27}z_{28}z_{35}^2t^{121}}{z_{23}^2z_{25}z_{26}^2z_{31}z_{33}z_{34}^2} & 0 & 0 \\ 0 & \frac{\zeta t^{184}}{z_{27}z_{30}z_{34}} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{z_{18}z_{24}z_{28}z_{32}t^{211}}{z_{21}z_{23}^2z_{25}z_{33}^3z_{35}} & z_0t^{163} & 0 & \frac{\zeta z_{34}t^{133}}{z_{25}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta z_{23}z_{25}z_{33}}{z_{24}z_{31}t^{146}} & \frac{\zeta^5 z_{31}z_{35}t^{184}}{z_{23}z_{27}^2z_{28}z_{30}z_{34}^2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^9 z_{30}}{z_{27}t^{269}} & 0 & 0 & \frac{\zeta^7}{t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^5 z_{27}t^{169}}{z_{30}z_{33}} & 0 & 0 & 0 \\ 0 & \frac{\zeta^9 z_{27}t^{184}}{z_{30}z_{34}} & \frac{z_{28}t^{136}}{z_{26}z_{34}z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^4 z_{19}z_{35}}{z_3z_{12}z_{13}z_{18}z_{32}t^{269}} & 0 & 0 & 0 & \frac{\zeta^4 z_{18}z_{21}z_{34}}{z_2z_{12}z_{25}z_{31}z_{32}z_{35}t^{17}} \\ \frac{\zeta z_{19}z_{33}z_{35}}{z_3z_{12}z_{13}z_{18}z_{32}t^{104}} & 0 & 0 & 0 & \frac{\zeta^9 z_{19}z_{26}z_{32}^2z_{33}z_{35}t^{148}}{z_3z_{12}z_{13}z_{18}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^4 z_2z_3z_{13}z_{31}}{z_{12}z_{19}z_{20}z_{21}z_{32}z_{34}t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{z_{34}t^{269}} & 0 & 0 & \frac{\zeta z_{24}z_{26}z_{35}}{z_{28}z_{34}t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^5}{t^{119}} & 0 & 0 & \frac{z_{24}z_{26}z_{35}t^{85}}{z_{28}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{\zeta^2 z_{15}z_{18}z_{19}z_{28}z_{32}t^{61}}{z_{21}z_{23}z_{25}z_{31}z_{33}^2z_{34}} & \frac{\zeta^{11}z_{15}z_{19}z_{35}t^{13}}{z_{34}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11}z_{15}z_{21}z_{23}z_{25}z_{31}z_{35}}{z_{18}z_{19}^2z_{26}z_{32}z_{34}t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{21}z_{23}z_{25}z_{33}z_{34}}{z_{15}^2z_{18}^2z_{19}^2z_{26}z_{31}^2z_{32}z_{35}^2t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^5 z_{25}z_{31}}{z_{23}z_{34}t^{269}} & 0 & 0 & 0 & \frac{\zeta^4}{t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}z_{25}z_{31}}{z_{23}t^{119}} & 0 & \frac{\zeta^7 z_{23}z_{24}z_{33}t^{163}}{z_{20}^2z_{31}} & 0 & \zeta^3 z_{34}t^{133} \\ 0 & \frac{\zeta^2 z_{23}^2z_{28}z_{34}^2t^{169}}{z_{25}^2z_{31}^2z_{33}z_{35}} & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^{10}z_{23}t^{136}}{z_{25}z_{31}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{11}z_3z_{19}}{z_{12}z_{21}z_{23}z_{25}z_{32}t^{269}} & 0 & 0 & 0 & \frac{\zeta^{10}z_2z_{18}z_{21}z_{23}z_{24}z_{34}}{z_{12}z_{13}z_{19}z_{20}z_{31}z_{32}z_{35}t^{17}} \\ \frac{\zeta^8 z_3z_{19}z_{33}}{z_{12}z_{21}z_{23}z_{25}z_{32}t^{104}} & 0 & 0 & 0 & \frac{\zeta^4 z_3z_{19}z_{26}z_{32}^2z_{33}t^{148}}{z_{12}z_{21}z_{23}z_{25}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{13}z_{31}z_{35}}{z_2z_3z_{12}z_{18}z_{24}z_{32}z_{34}t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^9 z_{28}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_{28}z_{34}}{t^{119}} & \frac{\zeta^{10}z_{34}t^{211}}{z_{28}z_{33}z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\zeta^5 z_{33}z_{35}t^{58}}{z_{34}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^5 z_{24}z_{26}z_{29}^3z_{30}}{z_{27}t^{269}} & 0 & \frac{\zeta^{11}z_{22}^3z_{27}z_{33}z_{35}t^{13}}{z_{20}z_{23}z_{28}z_{30}z_{31}^2} & \frac{\zeta^2}{t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}z_{24}z_{26}z_{29}^3z_{30}z_{34}}{z_{27}t^{119}} & 0 & 0 & \zeta z_{34}t^{85} & \frac{\zeta^{10}z_{18}z_{22}^3z_{26}z_{27}z_{32}z_{34}z_{35}t^{133}}{z_{20}z_{21}z_{23}^2z_{25}z_{28}z_{30}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}z_{18}z_{22}^3z_{25}z_{26}z_{27}z_{32}z_{33}z_{35}}{z_{20}z_{21}z_{23}z_{24}z_{28}z_{30}z_{31}t^{146}} & \frac{z_{27}t^{184}}{z_{24}z_{26}z_{29}^3z_{30}z_{34}} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^2 z_{25}z_{26}z_{31}z_{32}z_{35}}{z_{18}z_{21}z_{34}t^{269}} & \frac{\zeta^{11}z_{21}z_{24}z_{32}^2z_{35}t^{61}}{z_{25}z_{31}} & 0 & 0 & 0 \\ \frac{\zeta^{11}z_{25}z_{26}z_{31}z_{32}z_{33}z_{35}}{z_{18}z_{21}z_{34}t^{104}} & 0 & 0 & 0 & \frac{\zeta^{11}z_{28}t^{148}}{z_{21}^2z_{23}z_{24}z_{32}z_{33}z_{34}z_{35}^3} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11}z_{25}z_{31}z_{32}z_{35}}{z_{18}z_{21}z_{24}z_{34}t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\zeta^6 z_{18}z_{26}z_{31}z_{32}}{z_{20}z_{21}z_{34}t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta z_{21}^2z_{23}^2z_{24}z_{25}z_{33}}{z_{18}^2z_{19}^3z_{20}z_{26}^2z_{31}z_{32}^2t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11}}{t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^3 z_9z_{10}z_{11}z_{18}z_{19}z_{30}z_{32}z_{34}}{z_5z_7z_8z_{16}z_{21}z_{23}z_{25}z_{33}t^{269}} & 0 & 0 & 0 & \frac{\zeta z_5z_{11}z_{16}z_{18}z_{24}z_{26}z_{31}z_{32}}{z_4z_{20}z_{21}z_{25}t^{17}} \\ 0 & 0 & \frac{\zeta^{11}z_5z_{11}z_{16}z_{20}z_{33}t^{178}}{z_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}z_4}{z_{19}z_{20}z_{27}^3z_{30}z_{33}t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_4}{z_{19}z_{20}z_{27}z_{30}z_{34}t^{146}} & 0 & \frac{\zeta^6 z_9z_{10}z_{11}z_{18}z_{19}z_{27}z_{28}z_{32}t^{136}}{z_5z_7z_8z_{16}z_{21}z_{23}z_{25}z_{26}z_{33}z_{35}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^5 z_{28}z_{35}^2}{t^{269}} & 0 & 0 & \frac{\zeta^5}{t^{65}} & 0 \\ \frac{\zeta^2 z_{28}z_{33}z_{35}^2}{t^{104}} & 0 & 0 & \zeta^8 z_{33}t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{43} & 0 \\ 0 & 0 & 0 & \frac{\zeta^{10}z_{33}t^{58}}{z_{34}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{\zeta^9 z_{17}^2z_{21}z_{24}z_{28}z_{29}^2t^{61}}{z_{18}z_{22}^2z_{25}z_{27}^2z_{32}z_{33}z_{34}z_{35}} & \frac{\zeta^4 z_{22}z_{27}t^{13}}{z_{17}z_{26}z_{29}z_{31}^2} & \frac{\zeta z_{22}z_{27}z_{31}z_{33}z_{35}}{z_{17}z_{28}z_{29}t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^2 z_{17}^2z_{21}z_{24}z_{28}z_{29}^2t^{211}}{z_{18}z_{22}^2z_{25}z_{27}^2z_{32}z_{33}z_{35}} & 0 & \frac{z_{22}z_{27}z_{31}z_{33}z_{34}z_{35}t^{85}}{z_{17}z_{28}z_{29}} & \frac{\zeta^3 z_{18}z_{22}z_{27}z_{32}z_{34}t^{133}}{z_{17}z_{21}z_{23}z_{25}z_{29}z_{33}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^3 z_{18}z_{22}z_{25}z_{27}z_{32}}{z_{17}z_{21}z_{24}z_{29}z_{31}t^{146}} & \frac{\zeta^2 z_{17}^2z_{22}t^{184}}{z_{20}z_{23}z_{24}z_{29}z_{30}^2z_{31}z_{34}} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{11}z_{27}z_{28}z_{30}^2}{z_{26}z_{35}t^{269}} & 0 & 0 & 0 & \frac{\zeta^3}{t^{17}} \\ 0 & 0 & \frac{\zeta z_{20}^2 z_{21}z_{25}z_{33}t^{178}}{z_{18}z_{24}z_{26}z_{31}z_{32}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^7 z_{26}z_{35}t^{121}}{z_{27}z_{28}z_{30}^2 z_{33}} & 0 & 0 \\ 0 & 0 & \frac{z_{25}z_{31}t^{136}}{z_{23}z_{26}z_{34}^2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^8 z_{25}z_{26}z_{31}z_{32}z_{35}}{z_{18}z_{21}z_{34}t^{269}} & 0 & \frac{\zeta^4 z_{21}z_{24}z_{35}t^{13}}{z_{25}z_{26}z_{31}z_{32}} & 0 & 0 \\ \frac{\zeta^5 z_{25}z_{26}z_{31}z_{32}z_{33}z_{35}}{z_{18}z_{21}z_{34}t^{104}} & 0 & 0 & 0 & \frac{\zeta^5 z_{28}t^{148}}{z_{21}^2 z_{23}z_{24}z_{32}z_{33}z_{34}z_{35}^3} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^5 z_{25}z_{31}z_{32}z_{35}}{z_{18}z_{21}z_{24}z_{34}t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{10}z_5 z_{18}z_{19}z_{28}z_{30}z_{32}}{z_{21}z_{23}z_{25}z_{33}z_{35}t^{269}} & 0 & 0 & 0 & \frac{z_4 z_{11}z_{16}z_{31}}{z_5 z_{19}z_{20}t^{17}} \\ 0 & 0 & \frac{\zeta^{10}z_4 z_{11}z_{16}z_{20}z_{21}z_{25}z_{33}t^{178}}{z_5 z_{18}z_{19}z_{24}z_{26}z_{32}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}z_9 z_{10}z_{11}z_{18}z_{24}z_{26}z_{32}z_{34}z_{35}}{z_4 z_7 z_8 z_{16}z_{20}z_{21}z_{25}z_{27}z_{28}z_{30}z_{33}t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_9 z_{10}z_{11}z_{18}z_{24}z_{26}z_{32}z_{35}}{z_4 z_7 z_8 z_{16}z_{20}z_{21}z_{25}z_{27}z_{28}z_{30}t^{146}} & 0 & \frac{\zeta z_5 z_{18}z_{19}z_{27}^2 z_{28}^2 z_{32}t^{136}}{z_{21}z_{23}z_{25}z_{26}z_{33}z_{34}z_{35}^2} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^4 z_{17}^2 z_{22}z_{26}z_{29}}{z_{20}z_{23}z_{30}z_{34}t^{269}} & 0 & \frac{z_{22}z_{27}z_{28}z_{29}z_{30}z_{34}^2 t^{13}}{z_{17}z_{26}z_{31}z_{35}} & \frac{\zeta^8 z_{22}z_{24}z_{26}z_{29}z_{33}z_{35}^2}{z_{17}z_{27}z_{28}^2 z_{31}z_{34}t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{17}^2 z_{22}z_{26}z_{29}}{z_{20}z_{23}z_{30}t^{119}} & 0 & 0 & \frac{\zeta^7 z_{22}z_{24}z_{26}z_{29}z_{33}z_{35}^2 t^{85}}{z_{17}z_{27}z_{28}^2 z_{31}} & \frac{\zeta^{11}z_{18}z_{22}z_{27}z_{28}z_{29}z_{30}z_{31}z_{32}z_{34}^3 t^{133}}{z_{17}z_{21}z_{23}z_{25}z_{33}z_{35}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11}z_{18}z_{22}z_{25}z_{27}z_{28}z_{29}z_{30}z_{32}z_{34}^2}{z_{17}z_{21}z_{24}z_{35}t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \zeta^5 z_{19}z_{31}t^{13} & 0 & 0 \\ 0 & 0 & \zeta^8 z_{19}z_{31}z_{33}t^{178} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^8 z_{21}z_{23}z_{25}}{z_{18}z_{19}^2 z_{26}z_{31}z_{32}t^{161}} & 0 & 0 & 0 & 0 \\ \frac{z_{21}z_{23}z_{25}z_{33}}{z_{18}z_{19}^2 z_{26}z_{31}z_{32}z_{34}t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^3}{t^{269}} & \frac{\zeta^7 t^{61}}{z_{33}} & 0 & \frac{\zeta^{10}}{t^{65}} & 0 \\ \frac{z_{33}}{t^{104}} & 0 & 0 & \zeta z_{33}t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^4}{t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{\zeta^{10}z_{16}^2 z_{21}z_{23}z_{25}z_{28}t^{61}}{z_{18}z_{19}z_{32}z_{35}} & 0 & \frac{\zeta^4 z_{18}z_{19}^2 z_{26}z_{31}z_{32}z_{35}}{z_{16}z_{21}z_{23}z_{25}z_{28}t^{65}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta}{z_{16}z_{19}z_{26}z_{31}z_{33}t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^5}{z_{16}z_{19}z_{26}z_{31}z_{34}t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^7 z_{28}}{t^{269}} & \frac{\zeta^3 t^{61}}{z_{28}z_{33}z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{z_{28}z_{34}}{t^{119}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^4 t^{169}}{z_{28}z_{33}z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\zeta^3 z_{33}z_{35}t^{58}}{z_{34}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{z_{14}}{t^{269}} & \frac{\zeta^4 z_{30}t^{61}}{z_{14}^2 z_{27}z_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\zeta^{11}z_{14}z_{27}t^{43}}{z_{30}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^{10}z_{30}}{t^{269}} & \frac{\zeta^2 t^{61}}{z_{27}z_{30}z_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta z_{35}t^{121}}{z_{26}z_{27}z_{28}z_{30}z_{33}} & 0 & \frac{\zeta z_{27}z_{28}t^{91}}{z_{24}z_{35}} \\ 0 & 0 & 0 & 0 & \frac{\zeta^5 z_{27}z_{28}z_{33}t^{106}}{z_{24}z_{34}z_{35}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^4}{t^{269}} & \frac{\zeta^3 z_{28}t^{61}}{z_{33}z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_{34}}{t^{119}} & \frac{\zeta^2 z_{28}z_{34}t^{211}}{z_{33}z_{35}} & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^2 t^{121}}{z_{26}z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^4 z_{33}t^{106}}{z_{24}z_{34}} \end{pmatrix} \begin{pmatrix} \frac{\zeta^2 z_{14}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^9 z_{30}t^{226}}{z_{14}^2 z_{27}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta z_{30}t^{169}}{z_{14}^2 z_{27}z_{33}} & 0 & \frac{\zeta z_{14}z_{27}t^{43}}{z_{30}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^4 z_{30}}{t^{269}} & 0 & \frac{\zeta^4 z_{24}z_{35}t^{13}}{z_{27}z_{28}z_{30}z_{33}} & 0 & 0 \\ 0 & 0 & \frac{\zeta z_{24}z_{35}t^{178}}{z_{27}z_{28}z_{30}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^7 z_{27}z_{28}t^{91}}{z_{24}z_{35}} \\ 0 & 0 & 0 & 0 & \frac{\zeta^{11}z_{27}z_{28}z_{33}t^{106}}{z_{24}z_{34}z_{35}} \end{pmatrix} \begin{pmatrix} \frac{\zeta^8 z_{18}z_{22}z_{24}z_{26}z_{27}z_{32}}{z_{20}z_{21}z_{23}z_{31}z_{33}t^{269}} & 0 & \frac{\zeta^2 z_{20}z_{22}z_{27}z_{34}t^{13}}{z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta z_{18}z_{22}z_{24}z_{26}z_{27}z_{32}z_{34}}{z_{20}z_{21}z_{23}z_{31}z_{33}t^{119}} & 0 & \frac{\zeta^3 z_{21}z_{23}z_{24}z_{31}t^{163}}{z_{18}z_{22}^2 z_{26}z_{27}^2 z_{32}} & 0 & \frac{\zeta^{10}z_{22}z_{27}z_{31}z_{33}t^{133}}{z_{20}z_{23}z_{25}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}z_{22}z_{25}z_{27}z_{33}^2}{z_{20}z_{24}z_{34}t^{146}} & \frac{\zeta^3 z_{20}z_{22}z_{35}t^{184}}{z_{24}z_{27}z_{28}z_{30}} & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{10}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^3 z_{34}}{t^{119}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta^8 t^{121}}{z_{26}z_{33}} & 0 & 0 \\ 0 & z_1 t^{184} & \frac{\zeta^6 z_{25}z_{31}t^{136}}{z_{23}z_{34}^2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^5 z_{24}z_{25}z_{33}}{z_{20}^2 z_{34}t^{269}} & 0 & 0 & 0 & \frac{\zeta^8}{t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}z_{24}z_{25}z_{33}}{z_{20}^2 t^{119}} & 0 & \zeta^9 t^{163} & 0 & \frac{\zeta^3 z_{23}z_{24}z_{33}z_{34}t^{133}}{z_{20}^2 z_{31}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta}{t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^6}{t^{269}} & 0 & \frac{\zeta z_{24} t^{13}}{z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{34}}{t^{119}} & 0 & \frac{z_{24} z_{34} t^{163}}{z_{33}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^6 z_{33} t^{106}}{z_{24} z_{34}} \end{pmatrix} \begin{pmatrix} \frac{\zeta^{10} z_9 z_{34}}{t^{269}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta z_7 z_8 z_{11} z_{18} z_{31} z_{32} z_{35} t^{226}}{z_{10} z_{21} z_{23} z_{25} z_{27} z_{28} z_{30}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^5 z_9 z_{34}}{t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^9 z_9 z_{33}}{t^{146}} & 0 & \frac{\zeta z_9 z_{27} z_{28} t^{136}}{z_{26} z_{30} z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^7 z_{21} z_{24}}{z_{18} z_{23} z_{26} z_{32} t^{269}} & 0 & 0 & 0 & \frac{\zeta^5}{t^{17}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{z_{21} z_{24} z_{34}}{z_{18} z_{23} z_{26} z_{32} t^{119}} & 0 & 0 & 0 & \frac{\zeta^5 z_{21} z_{24} z_{34}^2 t^{133}}{z_{18} z_{25} z_{26} z_{31} z_{32}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^8 z_{18} z_{23} z_{25} z_{26} z_{32} z_{33}}{z_{20}^2 z_{21} z_{34} t^{146}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^9}{t^{269}} & 0 & 0 & \frac{\zeta^4}{t^{65}} & 0 \\ \frac{\zeta^6 z_{33}}{t^{104}} & 0 & 0 & \zeta^7 z_{33} t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10}}{t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{11} z_{25} z_{35}}{z_{13} t^{269}} & 0 & 0 & 0 & \frac{\zeta^7 z_{23} z_{34}}{z_{13} z_{31} z_{35}^2 t^{17}} \\ \frac{\zeta^8 z_{25} z_{33} z_{35}}{z_{13} t^{104}} & 0 & 0 & 0 & \frac{\zeta^4 z_{25} z_{26} z_{32}^3 z_{33} z_{35} t^{148}}{z_{13}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta^{11} z_{13}^2 z_{28} t^{169}}{z_{25}^2 z_{33}} & \frac{\zeta^9 z_{13}^2 z_{31} z_{35} t^{121}}{z_{23} z_{25} z_{33} z_{34}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^9 z_{25} z_{26} z_{31} z_{35}}{t^{269}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^6 z_{25} z_{26} z_{31} z_{33} z_{35}}{t^{104}} & 0 & 0 & 0 & \frac{\zeta^3 z_{18} z_{26} z_{32} t^{148}}{z_{21} z_{23} z_{33} z_{35}^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta^2 z_{23} z_{34} t^{91}}{z_{24} z_{35}^2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^9 z_7 z_{11} z_{16} z_{30} z_{34}}{z_6 z_{27} t^{269}} & 0 & 0 & \frac{\zeta^{11} z_6 z_{10} z_{21} z_{23} z_{25} z_{26} z_{30}}{z_9 z_{18} z_{32} t^{65}} & 0 \\ 0 & \frac{\zeta^2 z_8 z_{10} z_{16}^2 z_{26} z_{28} z_{33} z_{34} t^{226}}{z_{35}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^8 z_8 z_{10} z_{18} z_{32} z_{34}}{z_{16} z_{21} z_{23} z_{25} z_{31} z_{33} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{z_8 z_{10} z_{18} z_{32}}{z_{16} z_{21} z_{23} z_{25} z_{31} t^{146}} & 0 & \frac{z_7 z_{11} z_{16} z_{28} t^{136}}{z_6 z_{26} z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^{10} z_6 z_8 z_{10} z_{26} z_{30} z_{34}}{z_{27} t^{269}} & 0 & 0 & \frac{\zeta^2 z_{10} z_{16} z_{28} z_{30}}{z_6 z_9 z_{31} z_{35} t^{65}} & 0 \\ 0 & \frac{\zeta^4 z_7 z_{11} z_{16}^2 z_{21} z_{23} z_{25} z_{26} z_{31} z_{33} z_{34} t^{226}}{z_{18} z_{32}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{10} z_7 z_{11} z_{34} z_{35}}{z_{16} z_{28} z_{33} t^{161}} & 0 & 0 & 0 & 0 \\ \frac{\zeta^2 z_7 z_{11} z_{35}}{z_{16} z_{28} t^{146}} & 0 & \frac{\zeta z_6 z_8 z_{10} z_{28} t^{136}}{z_{35}} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\zeta^8}{z_{15}^2 z_{18} z_{21} z_{23} t^{269}} & \frac{\zeta^3 z_{15} z_{18} z_{28} t^{61}}{z_{23} z_{25} z_{26} z_{31} z_{32} z_{33}^2 z_{34} z_{35}} & \frac{\zeta^4 z_{15} z_{21} z_{24} z_{32} z_{35}^3 t^{13}}{z_{34}} & 0 & 0 \\ \frac{\zeta^5 z_{33}}{z_{15}^2 z_{18} z_{21} z_{23} t^{104}} & 0 & 0 & 0 & \frac{\zeta z_{26} z_{32}^3 z_{33} t^{148}}{z_{15}^2 z_{18} z_{21} z_{23}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\zeta^{11} z_{15} z_{23} z_{25} z_{31}}{z_{18} z_{21} z_{24} z_{34} t^{161}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\zeta^7 z_{28} z_{35}^2}{t^{269}} & \frac{t^{61}}{z_{28} z_{33} z_{35}^2} & 0 & \frac{\zeta^7}{t^{65}} & 0 \\ \frac{\zeta^4 z_{28} z_{33} z_{35}^2}{t^{104}} & 0 & 0 & \zeta^{10} z_{33} t^{100} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\zeta t^{169}}{z_{28} z_{33} z_{35}^2} & 0 & \zeta^2 t^{43} & 0 \\ 0 & 0 & 0 & \frac{z_{33} t^{58}}{z_{34}} & 0 \end{pmatrix}$$

We now give an overview of the method used to obtain the expressions for $\det_3$ and $T_{skewcw,4}^{\boxtimes 2}$:

Fix bases $a_i \in A$, $b_i \in B$, and $c_i \in C$. A tensor $T \in A \otimes B \otimes C$ has an expression $T = \sum_{i,j,k} T^{ijk} a_i \otimes b_j \otimes c_k$ and is *standard tight* in this basis if there exist injective functions $\omega_A : [m] \to \mathbb{Z}$, $\omega_B : [m] \to \mathbb{Z}$, $\omega_C : [m] \to \mathbb{Z}$ so that $T^{ijk} \neq 0$ implies $\omega_A(i) + \omega_B(j) + \omega_C(k) = 0$. In this case, we will call a choice of $(\omega_A, \omega_B, \omega_C)$ satisfying the constraints a set of *tight weights*. Given a set of tight weights for T, we consider border rank decompositions of the form:

$$T = \sum_{s=1}^r \mathcal{A}_s(t) \otimes \mathcal{B}_s(t) \otimes \mathcal{C}_s(t) + O(t), \tag{22}$$

where $\mathcal{A}_s(t) = \sum_{i=1}^m \mathcal{A}_{si} t^{\omega_A(i)} a_i$, $\mathcal{B}_s(t) = \sum_{j=1}^m \mathcal{B}_{sj} t^{\omega_B(j)} b_j$, and $\mathcal{C}_s(t) = \sum_{k=1}^m \mathcal{C}_{sk} t^{\omega_C(k)} c_k$. Note that when the tight weights are trivial, this is an ordinary rank decomposition. In our situation, the equations correspond to a strict subset of the equations describing a rank decomposition, namely those corresponding to triples (i,j,k) where $\omega_A(i) + \omega_B(j) + \omega_C(k) \leq 0$. In the case of $T_{skewcw,4}^{\boxtimes 2}$ this reduces the number of equations down from $\binom{25+2}{3} = 2925$ to 692 and just as with a rank decomposition, there are $3rm = 3150$ unknowns.

We pick a choice of tight weights which minimizes the number of equations to be solved. The problem of obtaining a border rank decomposition is then split into two questions: first, to compute a set of tight weights $(\omega_A, \omega_B, \omega_C)$ so that $\#\{(i,j,k) \mid \omega_A(i) + \omega_B(j) + \omega_C(k) \leq 0\}$ is minimal, and second, to solve the resulting equations (22) in the $\mathcal{A}_{si}$, $\mathcal{B}_{sj}$, $\mathcal{C}_{sk}$.

Consider the first question. Given sets $S_{\leq}, S_{>} \subset [m]\times[m]\times[m]$, consider the problem of deciding if there are tight weights $(\omega_A, \omega_B, \omega_C)$ satisfying the additional constraints that $\omega_A(i)+\omega_B(j)+\omega_C(k) \leq 0$ for $(i,j,k) \in S_{\leq}$ and $\omega_A(i)+\omega_B(j)+\omega_C(k) \geq 1$ for $(i,j,k) \in S_{>}$. These conditions along with the original equality conditions form a linear program on the images of $(\omega_A, \omega_B, \omega_C)$ which may be efficiently solved. There is no harm in letting the linear program be defined over the rationals, as we may clear denominators to obtain a solution in integers. One can use this fact to prune an exhaustive search of choices of $S_{\leq}, S_{>}$ to find one for which $S_{\leq} \cup S_{>} = [m]\times[m]\times[m]$, there exists a corresponding set of tight weights, and $\#S_{\leq}$ is minimal. While this is an exponential procedure, this optimization was sufficient to solve the problem for this decomposition.

The second problem, solving the associated system, is solved with the Levenberg-Marquardt nonlinear least squares algorithm [39, 41]. The sparse structure of the answer is obtained by speculatively zeroing (or setting to simple values) coefficients until all freedom with respect to the equations is lost. In other words, we impose additional simple equations on the solution and solve again until we obtain an isolated point, which can be verified by checking that the Jacobian has full rank numerically. This procedure is repeated many times in order to find a simple solution. Ideally, we would prove the resulting parameters indeed approximate an exact solution to the equations by searching for additional relations between the parameters and then using such relations to make symbolic methods tractable. In this case, all such attempts failed. See [18] for further discussion of these techniques.

The border rank decomposition in this section is also a Waring border rank decomposition, that is, $A = B = C$, and $\mathcal{A}_s(t) = \mathcal{B}_s(t) = \mathcal{C}_s(t)$; in particular, $\omega_A = \omega_B = \omega_C$. This condition was imposed to make the nonlinear search more tractable, and it also has independent interest. The techniques presented are equally applicable in the symmetric case as well as the asymmetric.

We remark that numerous relaxations of this method are possible. It was inspired by the improved expression for $\det_3$, which had the structure we assume. It remains to determine how useful it will be for more general types of tensors.

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