GEOMETRY AND REPRESENTATION THEORY IN THE STUDY OF MATRIX RIGIDITY

A Dissertation

by

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ABSTRACT

To be written

TABLE OF CONTENTS

LIST OF FIGURES

FIGURE Page

LIST OF TABLES

1. INTRODUCTION

The topic of matrix rigidity lies in the intersection of several areas of research in mathematics and computer science, including algebraic geometry, representation theory and algebraic complexity theory.

The origin of the subject is in a plan proposed by L. Valiant to prove lower bounds on the complexity of performing the Discrete Fourier Transform (DFT), that is the map $\mathbb{C}^n \to \mathbb{C}^n$ given by $\mathbf{x} \mapsto DFT_n\mathbf{x}$ where DFT_n is the $n \times n$ matrix $DFT_n = \left(\omega^{(j-1)(k-1)}\right)_{j,k}$ with $\omega = \exp(2\pi i/n)$. This linear map is of great importance in many areas of mathematics and computer science such as signal processing and data compression as well as representation theory and the study of differential equations. Its importance motivates the study of algorithms to perform the DFT matrix-vector multiplication using a small number of arithmetic operations.

In 1965, J.W. Cooley and J.W. Tukey ([CT65]) rediscovered an algorithm that was already known to Gauss to perform the DFT using $O(n\log(n))$ arithmetic operations (the so-called Fast Fourier Transform algorithm - FFT), whereas the standard multiplication algorithm uses $O(n^2)$ arithmetic operations. After the discovery of the FFT, people asked if it was possible to find an algorithm that was even faster. In 1977, L. Valiant ([Val77]) proposed an approach to determine lower bounds on the complexity of the DFT (and more generally of any linear map) based on the notion of *matrix rigidity*. The r-rigidity of a matrix A is the minimum number of entries of A that one needs to change so that the rank of the resulting matrix is at most r. More formally

Definition 1.1. Let A be an $n \times n$ matrix with complex coefficients and let r

be a nonnegative integer. The *r-rigidity* of A is the smallest integer s such that $A = B + C$, where $\operatorname{rank}(B) \leq r$ and C is s-sparse (namely it has at most s non-zero entries). More precisely, the *r-rigidity* of A is

$$\operatorname{Rig}_r(A) := \min\{s : A = B + C \text{ with } \operatorname{rank}(B) \leq r \text{ and } C \text{ is } s\text{-sparse}\}.$$

Intuition suggests that if a matrix A can be written as $A = B+C$ with B of low rank and C sparse, then the linear map $\mathbf{x} \mapsto A\mathbf{x}$ is easy to evaluate. For instance, using the standard multiplication algorithm, the evaluation of $\mathbf{x} \mapsto A\mathbf{x}$ requires $O(n^2)$ arithmetic operations. If A has rank $r < n$, then one can write $A = B_1B_2$, where B_1 is an $n \times r$ matrix and B_2 an $r \times n$ matrix; in this case the standard multiplication algorithm would require only $O(nr)$ arithmetic operations. On the other hand, if A is s-sparse, then the standard multiplication algorithm requires only $O(s)$ arithmetic operations.

Indeed, in [Val77], Valiant proved that the r-rigidity of A is a measure of the complexity of the linear map $\mathbf{x} \mapsto A\mathbf{x}$. The complexity measure that is used is expressed in terms of linear circuits. We refer to Appendix A for the precise definitions. The following result (and its refined version, see Theorem A.2), is our main motivation for the study of matrix rigidity.

Theorem 1.2 ([Val77], Cor. 6.3). *Let $\{A_n\}$ be a sequence of matrices with $A_n \in \operatorname{Mat}_n(\mathbb{C})$. If there exist $\varepsilon, \delta \in (0,1)$ such that $\operatorname{Rig}_{\varepsilon n}(A_n) \geq n^{1+\delta}$ then any linear circuit of logarithmic size depth computing A_n must have size $\Omega(n \log\log n)$.*

In his work, Valiant proposed to use Theorem 1.2 to prove lower bounds for the complexity of the DFT, or of any linear map. More generally, he posed the problem of finding any sequence of explicit matrices that would satisfy the hypotheses of Theorem 1.2:

Problem 1.3. Find an explicit infinite family of matrices A_n such that there exist $\varepsilon, \delta > 0$ with $\mathrm{Rig}_{\varepsilon n}(A_n) \geq n^{1+\delta}$.

In this context "explicit" has a precise meaning: we say that a sequence $\{A_n\}$ is *explicit* if the entries of A_n are computable by a deterministic Turing machine in time polynomial in n.

Problem 1.3, posed in 1977, is still open. More recently B. Barak (personal communication) proposed to restrict to the case where the matrix C of Definition 1.1 has the support of a permutation matrix:

Problem 1.4. Find an explicit infinite sequence of matrices $A_n \in \mathrm{Mat}_n$ with the property that there exists $\varepsilon \in (0,1)$ such that A_n cannot be written as $B_n + C_n$ if $\mathrm{rank}(B_n) \leq \varepsilon n$ and C_n has at most one non-zero entry on each row and each column.

We will see in Chapter 5 that Problem 1.4 has a simple solution if $\varepsilon < 1/2$. It becomes interesting when ε is close to 1: for instance, it is wide open if $\varepsilon \approx 9/10$.

A straightforward parameter count shows that if $\{A_n\}$ is a sequence of matrices with $A_n \in \mathrm{Mat}_n(\mathbb{C})$ generic, then $\mathrm{Rig}_{\varepsilon n}(A_n) \geq (1-\varepsilon)^2 n^2$ (we will see this argument in detail in Chapter 3); in particular, any sufficiently general sequence of matrices is a *non-explicit* solution for Problem 1.3. For this reason, questions such as Problems 1.3 and 1.4 in the computer science literature are associated to the expression *finding hay in a haystack*, referring to the fact that we seek an explicit object with some property in a set where almost every object has the required property. The expression was coined by Howard Karloff and is used in [AB09] in the context of explicit construction of expander graphs; another example is the problem of finding explicit examples of *elusive functions*, as discussed in [Raz08]. In our setting, in the haystack of all matrices, it is surprisingly easy to find the *needles*, namely the very rare matrices with low rigidity, and it is surprisingly hard to explicitly determine the *hay*, that is matrices with high rigidity.

In this thesis, following work started in [LTV03], [Lok06] and [KLPS14], we study the problem of matrix rigidity from a geometric point of view. Our goal is to determine sufficient conditions to have high rigidity and use these conditions to determine lower bounds for the rigidity of particular classes of matrices. This idea is common to many problems where algebraic geometry is used to determine lower bounds in complexity theory: the strategy consists in finding an algebraic variety that contains all the matrices with r-rigidity under some fixed threshold and determining equations for this algebraic variety; these equations are tests for high rigidity, in the sense that if a matrix A does not satisfy one of the equations, then we can conclude that its rigidity is higher than the fixed threshold. The same plan has been used, for instance, to determine lower bounds for the complexity of matrix multiplication and for the determinantal complexity of the permanent polynomial (see e.g. [Lan15]).

Part of this work is based on [GHIL16], of which we present the main results, while part is original. The main new achievement of this thesis is a method to determine equations providing tests for high rigidity, in a restricted range (in particular, in the setting of Problem 1.4) using a sequence of iterated determinants. We present some results that describe the equations we obtain in terms of the representation theory of the symmetric group that acts on Mat_n via conjugation by a permutation matrix. In Chapter 5, we use the methods we have developed to prove lower bounds on the rigidity of some class of matrices.

Previous work

We extensively explained the main contribution of [Val77]. Several results were achieved in this subject throughout the decades, mainly using combinatorial and graph-theoretic techniques; we refer to [Lok09] for a survey and to [Cod00] for a list of related problems. As for the problem of constructing explicit matrices with high rigidity, the state of the art is as follows. Over finite fields [Fri93] provides an explicit sequence of matrices A_n with $\mathrm{Rig}_r(A_n) = \Omega(\frac{n^2}{r}\log(\frac{n}{r}))$; a similar bound is provided by [SSS97] using a sequence of Cauchy matrices over infinite fields; the same bound is achieved in [Lok00] using the sequence DFT_n. Notice that in Valiant's range (namely with $r = \varepsilon n$ according to Theorem 1.2) this bound becomes the trivial $\Omega(n)$.

The use of geometry started essentially with [LTV03], where the first results translating matrix rigidity into a membership problem in algebraic geometry have been proved. With a similar approach, first [Lok06] and then [KLPS14] use effective upper bounds on the possible degrees of equations for certain varieties to construct rigid matrices: the solution of [Lok06] is obtained using primitive roots of one with order exponential in n, while [KLPS14] uses matrices whose entries are of the form $\sqrt{p_{jk}}$ (where p_{jk} are the first n^2 primes). In these cases the quadratic bound in Valiant's range is reached (namely $\mathrm{Rig}_{\varepsilon n}(A_n) = \Omega(n^2)$), but the matrices are not explicit in the sense explained above. In [GHIL16], we continued the study of matrix rigidity via geometry started in [LTV03] and [KLPS14]; most of these results are presented in detail in this thesis.

Structure of the thesis

In Chapter 2, we recall classical results in Algebraic Geometry and Representation Theory, that will be useful in the rest of the work. In particular, we describe in detail the construction of the Young symmetrizer associated to a standard Young tableau, that will be a fundamental tool in part of this work. In Chapter 3, we begin the study of the geometry of matrix rigidity: we define the notion of *border rigidity*, more suitable for geometry and we characterize the irreducible components of an algebraic variety that contains matrices of low border rigidity. These irreducible components are linear cones over determinantal varieties, classically studied objects in algebraic geometry. In Section 3.3, we present the results of [GHIL16] concerning the degrees of these linear cones. In Chapter 4, we determine several equations for the cones that we mentioned: these equations will be our tests for high rigidity, as explained above. In Section 4.1 we present some of the equations that we determined in [GHIL16] in the extreme cases $r = 1$ and $r = n - 2$. In Section 4.2, we focus on the restricted setting of 1.4: we provide a new method to determine equations via a sequence of iterated determinants in this restricted range; we prove the correctness of the method in Theorem 4.10. In Section 4.3, we study the sequence of iterated determinants from a representation theoretic point of view; Theorem 4.19 proves that some of the equations that we obtain are invariant under the action of a particular (rather natural) subgroup of the permutation group acting on the space of matrices by simultaneous permutation of rows and columns. In Chapter 5, we use some of the equations that we determined to prove that, in a restricted range, some sequences of matrices have high rigidity, namely they are hays in the haystack of all matrices. Finally, in Section 5.2, we give a specific result concerning Cauchy matrices in the setting of Problem 1.4 (Theorem 5.9). We conclude in Chapter 6, with a summary of the results, and a brief discussion on the difficulties we encountered and possible future research.

2. BACKGROUND IN ALGEBRAIC GEOMETRY AND REPRESENTATION THEORY

In this chapter we introduce basic notions of Algebraic Geometry and Representation Theory and we state the main classical results that will be needed in the rest of this work. We will work over the complex field $\mathbb{C}$.

2.1 Algebraic Geometry

Algebraic Geometry is the study of zero sets of polynomials. We will work both in affine and projective space. References for this section are [Har92], [Mum95] and [Sha77]. Let V be a finite dimensional vector space over $\mathbb{C}$. If U is a subset of V, we denote by $\langle U \rangle$ its linear span in V. We denote by $\mathbb{C}[V]$ the ring polynomials on V; if $v_1, \ldots, v_n$ is a basis of V and $x_1, \ldots, x_n$ is its dual basis in V^* then $\mathbb{C}[V]$ can be identified with the ring of polynomials in $x_1, \ldots, x_n$. We write $S^d V^*$ for the d-th symmetric power of V^*, or equivalently, in coordinates, the vector space of homogeneous polynomial of degree d in $x_1, \ldots, x_n$. If S is a set of polynomials, we denote by (S) the ideal generated by S in $\mathbb{C}[V]$. Given an ideal J, we denote by $\sqrt{J}$ its radical ideal.

Varieties and functions between them

Definition 2.1. An *affine algebraic variety* is a subset X in V that is the common zero set of a set of polynomials F in $\mathbb{C}[V]$:

$$X = \{x \in V : f(x) = 0 \text{ for every } f \in F\};$$

in this case, we write $Z(F) := X$.

We denote by $\mathbb{P}V$ the projective space of V, namely the space of 1-dimensional subspaces of V, or equivalently the quotient $(V \smallsetminus \{0\})/\sim$ where, for every $v_1, v_2 \in V \smallsetminus \{0\}$, $v_1 \sim v_2$ if and only if $v_1 = \lambda v_2$ for some $\lambda \in \mathbb{C}$. For $v \in V$, we denote by $[v]$ the class of v in $\mathbb{P}V$. If X is a subset of $\mathbb{P}V$, we write $\widehat{X}$ for the *affine cone* over X, namely $\widehat{X} := \{v \in V : [v] \in X\}$.

Given a homogeneous element $f \in \mathbb{C}[V]$, we say that f vanishes at $[v] \in \mathbb{P}V$ if f vanishes identically on the line $\langle v \rangle \subseteq V$.

Definition 2.2. A *projective algebraic variety* is a subset X in $\mathbb{P}V$ that is the common zero set of a set of homogeneous polynomials F in $\mathbb{C}[V]$. As in the affine case, we write $Z(F) := X$.

The family of affine algebraic varieties defines the closed sets of a topology on V, that is called *Zariski topology* of V. The Zariski topology is coarser than the Euclidean topology. Similarly, projective algebraic varieties define the closed sets of a topology in $\mathbb{P}V$, the Zariski topology on $\mathbb{P}V$. The Zariski topology of $\mathbb{P}V$ is the quotient of the Zariski topology of $V \smallsetminus \{0\}$ and it is coarser than the Euclidean topology on $\mathbb{P}V$.

We say that a property $\mathscr{P}$ holds generically in an algebraic variety X, or that it holds at a generic (or general) point of X, if there exists a Zariski open subset U in X such that $\mathscr{P}$ holds at every point of X.

Given $X \subseteq V$ (resp. $X \subseteq \mathbb{P}V$), let $I(X) := \{f \in \mathbb{C}[V] : f(x) = 0 \text{ for every } x \in X\}$. $I(X)$ is an ideal (resp. homogeneous ideal) of $\mathbb{C}[V]$ that is called the ideal of X. The quotient $\mathbb{C}[X] := \mathbb{C}[V]/I(X)$ is called the affine (resp. homogeneous) *coordinate ring* of X.

Theorem 2.3 (Nullstellensatz, [Mum95], Thm. 1.5)**.** *Let I be an ideal of $\mathbb{C}[V]$. Then $I(Z(I)) = \sqrt{I}$. In particular, $Z(I) = \emptyset$ in V if and only if $1 \in I$ and $Z(I) = \emptyset$ in $\mathbb{P}V$ if and only if $S^D V^* \subseteq I$ for some D sufficiently large..*

If $X \subseteq V$ is an affine variety, and $f \in \mathbb{C}[V]$ is a polynomial, f defines a function on X by restriction. The kernel of the restriction map is $I(X)$, therefore $\mathbb{C}[X]$ may be regarded as the ring of functions on X that can be obtained as restriction of polynomials on V. An element $\varphi \in \mathbb{C}[X]$ is called a *regular function* on X. Notice that if X is a projective variety, then $\mathbb{C}[X]$ is the ring of regular functions on $\widehat{X}$. The ideal $I(X)$ of a projective variety is a homogeneous ideal with respect to the natural grading of the polynomial ring; the grading descends to the homogeneous coordinate ring $\mathbb{C}[X]$, making it into a graded ring.

If X, Y are affine varieties and $Y \subseteq \mathbb{C}^m$, we say that $\Phi : X \to Y$ is a *regular map* if $\Phi = (\varphi_1, \dots, \varphi_m)$ if its components as a map to $\mathbb{C}^m$ are elements of $\mathbb{C}[X]$.

Similarly, if X, Y are projective varieties and $Y \subseteq \mathbb{P}\mathbb{C}^{(m+1)}$, we say that $\Phi : X \to Y$ is a *regular map* if its components as a map to $\mathbb{P}\mathbb{C}^{m+1}$ are elements of $\mathbb{C}[X]$ homogeneous, of the same degree and with no common zero set in X.

Regular maps define, via composition, a ring homomorphism between the coordinate rings; if $\Phi : X \to Y$ is a regular map of affine varieties then we can define a ring homomorphism as follows:

$$\begin{aligned} \Phi^* : \mathbb{C}[Y] &\to \mathbb{C}[X] \\ \varphi &\mapsto \varphi \circ \Phi. \end{aligned}$$

Proposition 2.4 ([Sha77], Thm. 1.10)**.** *Let $\Phi : X \to Y$ be a regular map of projec-*

tive varieties. Then the image $\Phi(X) \subseteq Y$ is a projective variety.

The map Φ^* sends equations for subvarieties of Y to equations for their preimage via Φ. More precisely

Lemma 2.5. *Let $\Phi : X \to Y$ be a regular map and let $Z \subseteq Y$ be a subvariety. Let $f \in \mathbb{C}[Y]$ such that f vanishes identically on Z. Then $\Phi^*(f)$ vanishes identically on $\Phi^{-1}(Z)$.*

Proof. Directly by the definition, if $z \in \Phi^{-1}(Z)$, then $\Phi^*(f)(z) = f \circ \Phi(z) = f(\Phi(z)) = 0$ since $\Phi(z) \in Z$. $\square$

Definition 2.6. Let X be an algebraic variety. We say that X is *reducible* if there exist algebraic varieties $Y_1, Y_2 \subsetneq X$ such that $X = Y_1 \cup Y_2$. We say that X is irreducible otherwise.

The following statement gives a sufficient condition for when Zariski and Euclidean closure coincide:

Proposition 2.7 ([Mum95], Thm. 2.33)**.** *Let $X \subseteq V$ (or $X \subseteq \mathbb{P}V$) and let $\overline{X}^Z$ and $\overline{X}^E$ be respectively its Zariski and its Euclidean closures in V (or in $\mathbb{P}V$). Then $\overline{X}^E \subseteq \overline{X}^Z$ and equality holds if $\overline{X}^Z$ is irreducible and X contains a Zariski open subset of $\overline{X}^Z$.*

Every algebraic variety can be expressed uniquely as finite irreduntant union of irreducible varieties (see e.g. [Sha77], Thm. 1.4 and Thm. 1.5). The irreducible varieties appearing in such union are called *irreducible components*. The ideal $I(X)$ of an irreducible (affine or projective) variety is a prime ideal, and its coordinate ring is an integral domain.

Definition 2.8. If X is an irreducible affine variety in V, define $\mathbb{C}(X) := \mathscr{Q}(\mathbb{C}[X])$

the field of fractions of $\mathbb{C}[X]$. The field $\mathbb{C}(X)$ is called the *field of rational functions* of the affine variety X. If X is projective, we define

$$\mathbb{C}(X) := \left\{ \frac{f}{g} \in \mathscr{Q}(\mathbb{C}[X]) : f, g \text{ are homogeneous of the same degree} \right\}.$$

$\mathbb{C}(X)$ is a field and it is called the *field of rational functions* of the projective variety X. Both in the affine and projective case, the elements of $\mathbb{C}(X)$ are called rational functions on X.

If X, Y are affine varieties and $Y \subseteq \mathbb{C}^m$, we say that $\Phi : X \dashrightarrow Y$ is a *rational map* if its components as a map to $\mathbb{C}^m$ are elements of $\mathbb{C}(X)$.

Similarly, if X, Y are projective varieties and $Y \subseteq \mathbb{P}\mathbb{C}^{(m+1)}$, we say that $\Phi : X \dashrightarrow Y$ is a *rational map* if its components as a map to $\mathbb{P}\mathbb{C}^{m+1}$ are elements of $\mathbb{C}(X)$, not all identically 0.

Definition 2.9. Let $\Phi : X \to Y$ be a rational map between projective irreducible varieties such that $\Phi^{-1}(y)$ is finite for a generic $y \in \Phi(X)$. Then there exists an integer d such that $d = |\Phi^{-1}(y)|$ for a generic $y \in \Phi(X)$ (see e.g. [Mum95], Prop. 3.17). The integer d is called *degree* of the map Φ; we write $\deg \Phi := d$.

Definition 2.10. Let $\dim W = n + 1$. The projective space $\mathbb{P}W$ can be covered by $n + 1$ copies of a n-dimensional affine spaces $V_0, \ldots, V_n$; we call them *affine charts* of $\mathbb{P}W$. If $X \subseteq V$ is an affine variety, we define $\overline{X}^{\mathbb{P}}$ to be its *projective completion*, namely $\overline{X}^{\mathbb{P}} := \overline{X} \subseteq \mathbb{P}W$, where W is an $n + 1$-dimensional vector space and V is regarded as one of the affine charts of $\mathbb{P}W$.

Dimension, degree, tangent space and tangent cone

A projective subspace of $\mathbb{P}V$ is the zero set of a set of linear forms; in particular, if $X \subseteq \mathbb{P}V$ is a subspace, then $\widehat{X} \subseteq V$ is a linear subspace of V. We write $X = \mathbb{P}^m$ if $\dim \widehat{X} = m + 1$.

Proposition 2.11 ([Har92], Prop. 11.4 and Prop. 7.16)**.** *Let $X \subseteq \mathbb{P}V$ be an irreducible projective variety. Then there exists a unique nonnegative integer c such that every every $\mathbb{P}^{c+1}$ intersects X in infinitely many points and at least one $\mathbb{P}^{c-1}$ does not intersect X. Moreover, there exists a unique integer d with the property that a generic $\mathbb{P}^c$ intersects X in exactly d points.*

Definition 2.12. Let $X \subseteq \mathbb{P}V$ be an irreducible variety. The *codimension* of X (in $\mathbb{P}V$) is the unique integer c such that every $\mathbb{P}^{c+1}$ intersects X in infinitely many points and at least one $\mathbb{P}^{c-1}$ does not intersect X. The *dimension* of X is $\dim \mathbb{P}V - c$. We write $\operatorname{codim} X := c$ and $\dim X := \dim \mathbb{P}V - c$. The dimension of an affine variety is defined to be the dimension of its projective completion. If X is reducible, then define $\dim X := \dim X_0$, where X_0 is an irreducible component with maximal dimension. We say that X is *equidimensional* if all its components have the same dimension.

Definition 2.13. Let $X \subseteq \mathbb{P}V$ be an irreducible variety. The *degree* of X is the unique integer d such that a generic $\mathbb{P}^c$ intersects X in d points, where $c = \operatorname{codim} X$. We write $\deg X := d$. The degree of an affine variety is defined to be the degree of its projective completion. If X is reducible, then define $\deg X = \sum_i \deg X_i$ where X_i's are the irreducible components of maximal dimension.

Remark 2.14. Let X be an algebraic variety (affine or projective). If $\operatorname{codim} X = 1$, then we say that X is a *hypersurface*. In this case $I(X)$ is a principal ideal (see e.g. [Sha77], Thm. 1.21); let $f \in \mathbb{P}V$ be a generator of $I(X)$. Then $\deg X = \deg(f)$.

Let $f \in \mathbb{C}[V]$ and let $v \in V$. Write $v = \sum_i c_i v_i$ where $c_i = x_i(v)$ are the coordinate of v with respect to the basis $v_1, \ldots, v_n$. Then f admits a Taylor expansion at v of the form

$$f = f_0 + f_1 + \cdots + f_d$$

where $d = \deg f$ and f_j is a homogeneous polynomial of degree j in the $(x_j - c_j)$'s. If α is a multi-index, the coefficient of a monomial $(\mathbf{x} - \mathbf{c})^\alpha$ in $f_{|\alpha|}$ is $\frac{1}{\binom{n}{\alpha}} \frac{\partial^{|\alpha|} f}{\partial \mathbf{x}^\alpha}(v)$. In particular $f_0 = f(v)$ and $f_1 = d_v f$, the differential of f at v.

Definition 2.15. Let X be an affine variety, let $x \in X$ and assume $I(X) = (g_1, \ldots, g_r)$. The *tangent space* to X at x is

$$T_x X := Z(d_x g_1, \ldots, d_x g_r).$$

Definition 2.16. Let X be an affine variety, let $x \in X$ and assume $I(X) = (g_1, \ldots, g_r)$. For every j write $g_j = g_{j,1} +, \ldots, + g_{j,d}$ for the Taylor expansion of g_j at x (since $x \in X$, we have $g_{j,0} = 0$). The *tangent cone* to X at x is

$$TC_x X := Z(g_{1,m_1}, \ldots, g_{r,m_r}),$$

where $m_j = \min\{\ell \in \{1, \ldots, d\} : g_{j,\ell} \neq 0\}$.

Definition 2.17. Let X be a projective variety and let $x \in X$. The *affine tangent space* to X at x is

$$\widehat{T}_x X := T_v \widehat{X}$$

where v is any non-zero vector in $\widehat{x}$ (the definition does not depend on the choice of v).

Similarly, the *affine tangent cone* to X at x is

$$\widehat{TC}_x X := TC_v \widehat{X}$$

where v is any non-zero vector in $\widehat{x}$ (the definition does not depend on the choice of v).

Notice that $\widehat{TC}_x X$ (and of course $\widehat{T}_x X$) is a cone over the line $\widehat{x}$. In particular, its image $\mathbb{P}\widehat{TC}_x X$ in the projective space $\mathbb{P}V$ (and similarly the image $\mathbb{P}\widehat{T}_x X$) defines a cone over the point x. This is called the *embedded* tangent cone to X at x (and the *embedded* tangent space to X at x). We denote them as $TC_x X$ and $T_x X$, respectively.

Definition 2.18. Let X be an affine (resp. projective) variety and let $x \in X$. We say that x is a *smooth* point of X if $\dim T_x X = \dim X$ (resp. $\dim \widehat{T}_x X = \dim X + 1$).

Proposition 2.19 ([Sha77], Thm. 2.3)**.** *Let X be an algebraic variety and let X_{smooth} be the set of its smooth points. Then X_{smooth} is Zariski open in X and at every smooth point the tangent space and the tangent cone coincide. Moreover the function $x \mapsto \dim T_x X$ (or $x \mapsto \dim \widehat{T}_x X - 1$) is upper semicontinuous and generically equal to* $\dim X$*. For every $x \in X$, if X is affine then* $\dim X = \dim TC_x X$*, if X is projective then* $\dim X = \dim \widehat{TC}_x X - 1$.

When it is irreducible and reduced, the (embedded) tangent cone to X at x can be interpreted as the union of tangent lines to X at x (see e.g. [Sha77], Sec. II.1.5.

2.2 Representation Theory

Representation Theory is the systematic study of symmetries or, more precisely, of group actions on vector spaces via linear transformation. We will focus on representation theory of finite groups and in particular of the symmetric group. References for this sections are [FH91], [Ser96] and [Sag13].

Definition 2.20. Let G be a group. A *representation* of G, or a G*-module*, is a vector space V with a group homomorphism

$$\rho_V : G \to GL(V);$$

we often drop ρ_V in our notation: for instance, if $v \in V$, gv will denote the image of v via $\rho_V(g)$.

Definition 2.21. Let G be a group and V a G-module. Let $X \subseteq V$ be a subset. We say that X is invariant under the action of G if $gx \in X$ for every $x \in X$ and $g \in G$. An algebraic variety that is invariant under the action of a group G is called a G*-variety*. We say that X is a set of G-invariants if $gx = x$ for every $x \in X$ and every $g \in G$.

Definition 2.22. Let G be a group and V a G-module. We say that V is irreducible if $V \neq 0$ and there is no (non-trivial) invariant subspace in V under the action of G.

Definition 2.23. Let G be a group and let V, W be representations of G. A linear map $f : V \to W$ is called G*-equivariant*, or a G*-map*, if it commutes with the action of G namely, for every $g \in G$ and every $v \in V$, $f(g \cdot v) = g \cdot f(v)$.

The following elementary result is an extremely useful tool in representation theory (we refer to [FH91], Lemma 1.7 for the proof):

Lemma 2.24 (Schur's Lemma)**.** *Let G be a group and let V, W be irreducible representations of G, let $f : V \to W$ be a G-equivariant map. Then either $f \equiv 0$ or f is an isomorphism. Moreover, if $V = W$, then $f = c \cdot \mathrm{id}_V$ for some $c \in \mathbb{C}$.*

Definition 2.25. Let V be a representation of a group G and let V_0 be an irreducible representation of G. We say that V_0 has multiplicity k in V if k is the largest integer such that there exists an injective G-equivariant map $V_0^{\oplus k} \to V$.

Theorem 2.26 ([Ser96], Thm. 2 and Cor. 1)**.** *Let G be a finite group. Then every representation V of G is direct sum of irreducible representations:*

$$V = V_1^{\oplus k_1} \oplus \cdots \oplus V_\ell^{\oplus k_\ell},$$

with V_i non-isomorphic irreducible representations. The multiplicities k_i are uniquely determined.

Definition 2.27. Let G be a finite group. The *group algebra* $\mathbb{C}[G]$ is the $|G|$-dimensional vector space of $\mathbb{C}$-valued functions on G. For every $g \in G$, let $\delta_g : G \to \mathbb{C}$ be the function defined by

$$\delta_g(h) := \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{otherwise} \end{cases}$$

The δ_g's form a basis of $\mathbb{C}[G]$ and the multiplication structure is extended linearly from the product in G: $\delta_{g_1} \cdot \delta_{g_2} = \delta_{g_1 g_2}$.

Every representation of G is naturally a module over the ring $\mathbb{C}[G]$ (see e.g. [Ser96] for details); an irreducible G-representation defines a $\mathbb{C}[G]$-module that has not proper submodules; a G-equivariant map is a $\mathbb{C}[G]$-linear map of modules. In particular, *the representation theory of G is equivalent to the theory of $\mathbb{C}[G]$-modules.*

The group G acts on $\mathbb{C}[G]$ in two ways: via multiplication and via pull-back of functions. More precisely, $\mathbb{C}[G]$ is acted on by $G \times G$ as follows:

$$(g_1, g_2) \cdot \delta_h(-) := \delta_{g_1^{-1}} \cdot \delta_h(g_2 \cdot -) = \delta_{g_1^{-1} h g_2}(-),$$

and the action is extended linearly to $\mathbb{C}[G]$.

The following result describes the irreducible representations of a finite group:

Theorem 2.28 ([Ser96], Ch.2, Thm. 7)**.** *Let G be a finite group. Then there is a one-to-one correspondence between the conjugacy classes of G and the irreducible representations of G. Moreover, if $V_1, \ldots, V_r$ are the irreducible representations of G, then the group algebra $\mathbb{C}[G]$ decomposes as*

$$\mathbb{C}[G] = \bigoplus_{i=1}^{r} V_i \otimes V_i^*. \tag{2.1}$$

It is straightforward to verify (see e.g. [Ser96], Ch.2) that the group algebra $\mathbb{C}[G]$ acts on a module V by contracting the irreducible components of V on the second factors of the decomposition (2.1); more precisely, if $V = V_j$ is irreducible, and $g = u \otimes \alpha \in V_i \otimes V_i^* \subseteq \mathbb{C}[G]$, then $g : V \to V$ is defined by $g \cdot v = \alpha(v)u$, where the elements of V_i^* are identically 0 on V_j if $j \neq i$ and they have a natural contraction on V_j if $j = i$.

In the next section, we will present in detail the irreducible representations of the symmetric group and we define projection operators (so-called Young symmetrizers) that are useful to construct explicitly the irreducible representations.

Specht modules and Young symmetrizers

Let $\mathfrak{S}_d$ denote the *symmetric group* over $\{1, \ldots, d\}$. We will represent permutations in a two-rows notation or as product of disjoint cycles. For instance

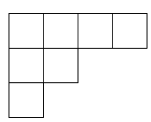
$$\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}\right) = (1,2)(3,4)$$

denotes the permutation of $\mathfrak{S}_4$ defined by the bijective map

$$\sigma(1) = 2, \quad \sigma(2) = 1, \quad \sigma(3) = 4, \quad \sigma(4) = 3.$$

The *sign* of a permutation σ is $+1$ if σ can be written as product of an even number of 2-cycles and it is -1 if σ can be written as an odd number of 2-cycles. We denote by $(-1)^\sigma$ the sign of the permutation σ.

A *partition* is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The integer $|\lambda| := \sum_{i=1}^{\ell}$ is called the order (or the number of boxes) of λ and $\ell(\lambda) := \ell$ is called the length (or the number of parts) of λ. If $|\lambda| = d$ and $\ell(\lambda) = \ell$, we say that λ is a partition of d with ℓ parts, and we write $\lambda \vdash_{\overline{\ell}} d$. It is useful to represent partitions via *Young diagrams* which are top-left justified collections of boxes. The partition $\lambda \vdash_{\overline{\ell}} d$ is represented by a Young diagram with d boxes, where the i-th row has λ_i boxes. For example, the diagram



represents the partition $(4, 2, 1)$ of 7.

Every permutation $\sigma \in \mathfrak{S}_d$ decomposes uniquely as product of disjoint cycles: we say that σ has *cycle type* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ if the cycles in its decomposition have lengths $\lambda_1 \geq \cdots \geq \lambda_\ell$. Two permutations $\sigma_1, \sigma_2 \in \mathfrak{S}_n$ are conjugate in $\mathfrak{S}_d$ if and only if they have the same cycle type. In particular, conjugacy classes in $\mathfrak{S}_d$ are indexed by partitions of d and so are the irreducible representations of $\mathfrak{S}_d$ by Theorem 2.28. Denote by $[\lambda]$ the irreducible representation corresponding to the partition λ: it is called the *Specht module* of type λ.

Even though $[\lambda]^* \simeq [\lambda]$ (see e.g. [Ike12], Sec. 4.1), in order to avoid confusion, we tend to denote explicitly with $[\lambda]^*$ the modules appearing on the right hand side of (2.1).

A *Young tableau* is a function that assigns a positive integer to each box of a Young diagram. Pictorially, it corresponds to a filling of the Young diagram with positive integers. If a Young tableau T_λ is a filling of a Young diagram λ, we say that T_λ has *shape* λ. The content of a Young tableaux is the function that counts how many times each integer appears in a filling.

There is a natural action of $\mathfrak{S}_d$ on the set of all Young tableaux, that permutes the integers appearing in each box.

A *standard* Young tableau of shape λ is a Young tableau of content $(1^{|\lambda|})$ that is increasing in every row from left to right and in every column from top to bottom.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 3 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 4 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Figure 2.1: A standard and a non-standard Young tableaux of shape $(4, 2, 1)$

Every standard tableaux has an associated Young symmetrizer, a particular element of the group algebra $\mathbb{C}[\mathfrak{S}_d]$ that defines a projection operator on the Specht modules. We follow [FH91], §4, for the construction of the Young symmetrizers.

Let T_λ be a standard Young tableau of shape $\lambda \vdash_{\ell} d$. Define:

$$R_{T_\lambda} = \{\sigma \in \mathfrak{S}_d : \sigma \cdot T_\lambda \text{ has the same rows (as sets) as } T_\lambda\},$$
$$C_{T_\lambda} = \{\tau \in \mathfrak{S}_d : \tau \cdot T_\lambda \text{ has the same columns (as sets) as } T_\lambda\}.$$

Respectively, we call R_{T_λ} and C_{T_λ} the *stabilizer of the rows* and *the stabilizer of the columns* of T_λ.

Define two elements of $\mathbb{C}[\mathfrak{S}_d]$ as follows:

$$a_{T_\lambda} := \sum_{R_{T_\lambda}} \sigma, \qquad b_{T_\lambda} := \sum_{C_{T_\lambda}} (-1)^\tau \tau;$$

they are called, respectively, the *symmetrizer of the rows* and *the symmetrizer of the columns* of T_λ.

The *Young symmetrizer* associated to T_λ is the element $\mathcal{Y}_{T_\lambda} = b_\lambda a_\lambda \in \mathbb{C}[\mathfrak{S}_d]$. The importance of this element is due to the following theorem:

Theorem 2.29 (see e.g. [FH91], Thm 4.3)**.** *The right ideal $\mathcal{Y}_{T_\lambda}\mathbb{C}[\mathcal{S}_d]$ is isomorphic to $[\lambda]^* \simeq [\lambda]$, as right-$\mathbb{C}[G]$-module: more precisely, left multiplication by $\mathcal{Y}_{T_\lambda}$ defines a $\mathfrak{S}_d$-equivariant projection*

$$\mathbb{C}[\mathfrak{S}_d] \to v_\lambda \otimes [\lambda]^*,$$

for some $v_\lambda \in [\lambda]$.

More generally, if V is a $\mathfrak{S}_d$-representation, then $\mathcal{Y}_{T_\lambda}$ can be regarded as a projection operator in V whose image is contained in the subspace $[\lambda]^{\oplus k_\lambda}$ (where k_λ is the multiplicity of $[\lambda]$ in V. More precisely, we have the following results

Lemma 2.30. *Let $\lambda, \mu \vdash d$ be two partitions and let $\mathcal{Y}_{T_\lambda}$ be the Young symmetrizer of a Young tableau T_λ of shape λ. Then*

$$\mathcal{Y}_{T_\lambda}[\mu] = \begin{cases} \langle v_\lambda \rangle & \text{for some non-zero } v_\lambda \in [\mu] \text{ if } \mu = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

We can exploit the projection defined by $\mathcal{Y}_{T_\lambda} \in \mathrm{End}_{\mathfrak{S}_d}(V)$ to detect the multiplicity of $[\lambda]$ in a representation V of $\mathfrak{S}_d$.

Corollary 2.31. *Let T_λ be a Young tableau of shape λ and let $\mathcal{Y}_{T_\lambda}$ be its Young symmetrizer. Let $V = \bigoplus_\mu [\mu]^{\oplus k_\mu}$ be a $\mathfrak{S}_d$-representation. Then $\mathcal{Y}_{T_\lambda}$ defines a projection $V \to V$ whose image is a space of dimension k_λ.*

Indeed, Young symmetrizers of a given shape provide a basis of the corresponding Specht module. If $T_1, \dots, T_N$ are all the standard Young tableaux of shape λ, and $\mathcal{Y}_1, \dots, \mathcal{Y}_N$ are the corresponding Young symmetrizers, let v_i be a non-zero vector in the one-dimensional space $\mathcal{Y}_i[\lambda]$. We have that $\{v_i : i = 1, \dots, N\}$ is a basis for $[\lambda]$, that we call *the basis of standard Young tableaux of shape* $[\lambda]$. In particular, the dimension of $[\lambda]$ is the number of standard Young tableaux of shape λ.

The action of $\mathfrak{S}_d$ on the set $\mathcal{T}_\lambda$ of Young tableaux of shape λ and content $(1^{|\lambda|})$ provides another interpretation of the Specht module $[\lambda]$. Let $V_{\mathcal{T}_\lambda}$ be the vector space of linear combinations of elements of $\mathcal{T}_\lambda$; for $T \in \mathcal{T}_\lambda$, write v_T for the basis vector corresponding to $\mathcal{T}_\lambda$.

We define three type of *straightening operations* or *Garnir operations*:

- substitute v_{T_1} with $-v_{T_2}$ if T_1, T_2 differ only in one column by a single transposition of two entries;

- substitute v_{T_1} with v_{T_2} if T_1, T_2 differ only by a transposition of two columns of the same length;
- substitute v_T with $\sum_{T'} v_{T'}$ where the sum is over all T' that can be obtained from T by exchanging, for fixed j, k, the top k elements of column $j+1$ with any k elements of the column j.

Consider the subspace $K(\lambda) \subseteq V_{\mathcal{T}_\lambda}$, generated by differences $w_2 - w_1$ of elements of $V_{\mathcal{T}_\lambda}$ with the property that w_2 can be obtained from w_1 with a finite number of straightening operations. Then the action of $\mathfrak{S}_d$ passes to the quotient $V_{\mathcal{T}_\lambda}/K(\lambda)$ and there is an isomorphism of $\mathfrak{S}_d$-representations $[\lambda] = V_{\mathcal{T}_\lambda}/K(\lambda)$. If $\mathcal{Y}_T$ is the Young symmetrizer of $T \in \mathcal{T}_\lambda$, then $\mathcal{Y}_T(V_{\mathcal{T}_\lambda}/K) = \langle v_T \rangle$. We refer to Sec. 4.1 in [Ike12] and Sec. 8.1 in [Ful97] for details on this construction and generalizations.

Induced representations

If H is a subgroup of G, then every representation W of H defines naturally a representation of G, that we denote $\mathrm{Ind}_H^G(W)$, the *induced representation* of W from H to G. In this section, we discuss induced representations, with a focus on the symmetric group.

First, we construct $\mathrm{Ind}_H^G(W)$ explicitly. Let $k := |G : H|$ be the index of H in G and let $\mathbf{g} := \{g_1, \dots, g_k\}$ be a set of representatives for the left cosets of H in G, namely $g_1 H, \dots, g_k H$ are the cosets of H in G.

Let $M_{\mathbf{g}} = \langle g_1, \dots, g_k \rangle$ be a vector space having $\mathbf{g}$ as basis.

We define functions $h(\cdot)$ and $j(\cdot)$ as follows: for g in G, write $h(g) \in H$ and $j(g) = 1, \dots, k$ for the unique elements such that $g = g_{j(g)} h(g)$.

Then define $\mathrm{Ind}_H^G(W)$ as $M_{\mathbf{g}} \otimes W$ with the left G-action given on basis elements as follows and extended by linearity: for every $g \in G$, $g_i \in \mathbf{g}$, $w \in W$, set:

$$g \cdot (g_i \otimes w) = g_{j(gg_i)} \otimes h(gg_i)w.$$

It is straightforward to verify that the action is well defined. Moreover, we can see that (the isomorphism class of) $\mathrm{Ind}_H^G(W)$ does not depend on the set of representatives $\mathbf{g}$.

In terms of group algebras, we have $\mathrm{Ind}_H^G(W) = W \otimes_{\mathbb{C}[H]} \mathbb{C}[G]$, where $\mathbb{C}[H]$ is regarded as a subring of $\mathbb{C}[G]$ and therefore $\mathbb{C}[G]$ is a $\mathbb{C}[H]$-module.

In Chapter 4, we will be particularly interested in the induced representations from $\mathfrak{S}_d \times \mathfrak{S}_e$ to $\mathfrak{S}_{d+e}$. Let μ and ν be partitions of d and e respectively, so that $[\mu] \otimes [\nu]$ is an irreducible representation of $\mathfrak{S}_d \times \mathfrak{S}_e$: then

$$\mathrm{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_e}^{\mathfrak{S}_{d+e}}([\mu] \otimes [\nu]) = \bigoplus_{\lambda \vdash d+e} [\lambda]^{\oplus c_{\mu,\nu}^{\lambda}}$$

where $c_{\mu,\nu}^{\lambda}$ are nonnegative integers called *Littlewood-Richardson coefficients*.

Several combinatorial techniques to compute Littlewood-Richardson coefficients are known; we refer to [Ike12] for an extensive discussion. We will use Littlewood-Richardson coefficients only in a very particular case:

Lemma 2.32 (Pieri's Rule [Mac98], I.5)**.** *Let μ be a partition of e and let $\nu = (d)$*

and $\pi = (1^d)$. *Then*

$$c^{\lambda}_{\mu,\nu} = \begin{cases} 1 & \begin{array}{c}\textit{if } \lambda \textit{ can be obtained from } \mu \textit{ by adding } d \textit{ boxes} \\ \textit{no two of them in the same column,}\end{array} \\ 0 & \textit{otherwise,} \end{cases}$$

$$c^{\lambda}_{\mu,\pi} = \begin{cases} 1 & \begin{array}{c}\textit{if } \lambda \textit{ can be obtained from } \mu \textit{ by adding } d \textit{ boxes} \\ \textit{no two of them in the same row,}\end{array} \\ 0 & \textit{otherwise.} \end{cases}$$

3. GEOMETRY OF MATRIX RIGIDITY

In this chapter, we explain how algebraic geometry plays a role in the study of matrix rigidity. In particular, we will show how algebraic geometry will provide sufficient conditions for high rigidity. We will introduce the notion of border-rigidity, more suitable for geometry and we will determine the irreducible decomposition of the variety of matrices of low border rigidity. We will give a formula for the degree of these components in a restricted case.

The main reference for this chapter is [GHIL16].

We introduce some notation and some basic notions that will be useful. We denote by Mat_n the vector space of $n \times n$ matrices with complex coefficients. The polynomial ring $\mathbb{C}[\mathrm{Mat}_n]$ is generated by variables x^i_j, for $i, j = 1, \dots, n$, where $x^i_j \in \mathrm{Mat}^*_n$ outputs the (i, j)-th entry of a matrix.

Given two sets of indices $I, J \subseteq \{1, \dots, n\}$ with the same cardinality $|I| = |J| = k$, we write M^I_J for the minor of the matrix $\left(x^i_j\right)_{i,j=1,\dots,n}$ obtained by considering the rows in I and the columns in J. $M^I_J \in \mathbb{C}[\mathrm{Mat}_n]$ is a homogeneous polynomial of degree k in the x^i_j's.

If $S \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$ is a set of pairs, we write $L^S \subseteq \mathrm{Mat}_n$ for the linear space of matrices supported at the entries of S, namely

$$L^S = \{A \in \mathrm{Mat}_n : A^i_j = 0 \text{ if } (i, j) \in S\}.$$

It will be useful to represent S pictorially by marking with ✘ the entries in S. For instance, if $n = 3$ and $S = \{(1,1),(1,2),(2,1)\}$, we represent it as

$$\begin{pmatrix} \text{✘} & \text{✘} & \\ \text{✘} & & \\ & & \end{pmatrix}.$$

Sometimes, it will be useful to represent the entries that are <u>not</u> in S. In this case we use the symbol $\Diamond$. For instance, if S consists of all the entries of a 3×3 matrix except $\{(1,1),(1,2),(2,1)\}$, then we represent it as

$$\begin{pmatrix} \Diamond & \Diamond & \\ \Diamond & & \\ & & \end{pmatrix}.$$

3.1 Algebraic geometry useful for matrix rigidity

Joins of algebraic varieties

Given p, q two points of $\mathbb{P}V$, we denote by $\langle p, q\rangle := \mathbb{P}(\widehat{p} + \widehat{q})$ the projective span of p and q. The span $\langle p, q\rangle$ is a $\mathbb{P}^1$ if p and q are distinct (and it coincides with the projective line joining them) and it is the point p if $p = q$.

Definition 3.1. Let $X, Y \subseteq \mathbb{P}V$ be projective varieties. The *join* of X and Y is

$$J(X,Y) := \overline{\bigcup_{x\in X, y\in Y} \langle x, y\rangle}.$$

It is possible to give an upper bound for the dimension of $J(X,Y)$ in terms of the dimension of X and Y.

Proposition 3.2. *Let $X, Y \subseteq \mathbb{P}V$ be irreducible projective varieties. Then $J(X,Y)$ is irreducible and* $\dim J(X,Y) \leq \dim X + \dim Y + 1$.

Proof. Define

$$\mathcal{J}^0(X,Y) := \Big\{(x,y,p) \in X \times Y \times \mathbb{P}V : x \in X, y \in Y, x \neq y, p \in \langle x, y \rangle\Big\},$$

and $\mathcal{J}(X,Y) := \overline{\mathcal{J}^0(X,Y)}$, the closure in the Zariski topology of $X \times Y \times \mathbb{P}V$. Consider the two projections

$$\begin{tikzcd} & \mathcal{J}(X,Y) \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ X \times Y & & \mathbb{P}V \end{tikzcd}$$

The projection π_1 surjects onto $X \times Y$ and the image of π_2 is $J(X,Y)$.

Consider the restriction of the projections π_1, π_2 to $\mathcal{J}^0(X,Y)$: π_1 surjects onto $(X \times Y) \smallsetminus \Delta_{(X \times Y)}$, where $\Delta_{(X \times Y)} = \{(x,y) \in X \times Y : x = y\}$, that is a Zariski open subset of $X \times Y$; on the other hand π_2 surjects onto a Zariski dense subset of $J(X,Y)$.

The fiber of the projection π_1 over $(x,y) \in (X \times Y) \smallsetminus \Delta_{(X \times Y)}$ is the line $\langle x, y \rangle$. In particular all fibers are irreducible and of dimension 1. By Thm. 1.25 in [Sha77], we deduce that $\mathcal{J}(X,Y)$ is irreducible and that its dimension is $\dim(X \times Y) + \dim\langle x, y \rangle = \dim X + \dim Y + 1$.

Since π_2 surjects onto $J(X,Y)$, we conclude $\dim J(X,Y) \leq \dim X + \dim Y + 1$. $\square$

For two projective varieties $X, Y \subseteq \mathbb{P}V$, we say that $J(X,Y)$ has the *expected di-*

mension if $\dim J(X,Y) = \min\{\dim X + \dim Y + 1, \dim \mathbb{P}V\}$.

If $\widehat{X}, \widehat{Y} \subseteq V$ are cones over two projective varieties $X, Y \subseteq \mathbb{P}V$, we define $J(\widehat{X}, \widehat{Y}) := \widehat{J}(X,Y)$. Notice that $J(\widehat{X}, \widehat{Y}) = \overline{\{x + y \in V : x \in \widehat{X}, y \in \widehat{Y}\}}$. The join $J(X,Y)$ has the expected dimension if and only if $\dim(\widehat{X}, \widehat{Y}) = \dim(\widehat{X}) + \dim(\widehat{Y})$ and in this case we will say that $J(\widehat{X}, \widehat{Y})$ has the expected dimension.

Joins between an algebraic variety and a linear space can be obtained as preimages of projection maps; we characterize them in the following result, whose proof is immediate.

Lemma 3.3. *Let $X \subseteq \mathbb{P}V$ be a projective algebraic variety and let $L \subseteq V$ be a linear subspace. Let $\pi : V \to V/L$ be the projection map. Then $J(\widehat{X}, L) = \pi^{-1}(\overline{\pi(\widehat{X})})$.*

Determinantal varieties

Definition 3.4. Given a nonnegative integer $r \leq n$, define $\sigma_r^{(n)}$ to be the set of $n \times n$ matrices of rank at most r, that is

$$\sigma_r^{(n)} := \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) : \mathrm{rank}(A) \leq r\}.$$

The set $\sigma_r^{(n)}$ is an affine algebraic variety and it is called the *general r-th determinantal variety* of $n \times n$ matrices. We drop the superscript if the size of the matrices is clear from the contest.

The ideal of $\sigma_r^{(n)}$ is generated by minors of size $(r+1)$ (see e.g. [ACGH85], Sec. II.3), namely

$$I(\sigma_r) = \left(M_J^I : |I| = |J| = r+1\right).$$

In particular, σ_r is the affine cone over a projective variety. We have $\operatorname{codim} \sigma_r =$

$(n-r)^2$ and $\dim \sigma_r = r(2n-r)$.

The variety σ_r is irreducible. It is singular and its singular locus coincides with $\sigma_{r-1} \subseteq \sigma_r$. In particular, the subset of σ_r consisting of matrices of rank exactly r is the set of the smooth points of σ_r. The group $GL_n \times GL_n$ acts on Mat_n, via $(g,h)A := g^{-1}Ah$ for every $g, h \in GL_n$ and $A \in \mathrm{Mat}_n$. For every r, let $X_r = \left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where I_r is the $r \times r$ identity matrix and the blocking is $(r, n-r) \times (r, n-r)$. The orbit of X_r under the action of $GL_n \times GL_n$ is the set of matrices of rank exactly r. The variety σ_r is the closure of the orbit of X_r.

Proposition 3.5 (see e.g. [Har92], Example 14.16 and Example 20.5)**.** *Let* $A \in \sigma_r$ *with* $\mathrm{rank}(A) = \ell \leq r$. *Then*

$$\begin{aligned} T_A\sigma_r &= \{B \in \mathrm{Mat}_n : B \cdot \ker A \subseteq \mathrm{Im}\, A\}, \\ TC_A\sigma_r &= \{B \in \mathrm{Mat}_n : \ker A \xrightarrow{B} \mathbb{C}^n \to (\mathbb{C}^n/\mathrm{Im}\, A) \textit{ has rank at most } k-\ell\}. \end{aligned}$$

In particular, if $A = X_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (with blocking $(1, n-1) \times (1, n-1)$), then

$$TC_A\sigma_r = J(T_A\sigma_1, \tau_r)$$

where $\tau_r := \left\{ B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & B' \end{smallmatrix}\right) : B' \in \sigma_r^{(n-1)} \right\}$.

The degree of σ_r is (see [ACGH85], Sec. II.5)

$$\deg \sigma_r = \prod_{j=0}^{n-r-1} \frac{(n+j)!j!}{(r+j)!(n-r+j)!} \tag{3.1}$$

and the degree of its tangent cone at A, with $\mathrm{rank}(A) = \ell$ is

$$\deg TC_A \sigma_r = \deg \sigma_{r-\ell}^{(n-\ell)} = \prod_{j=0}^{n-r-1} \frac{(n-\ell+j)!j!}{(r-\ell+j)!(n-r+j)!}.$$

3.2 Variety of matrices of low rigidity

In this section, we define an algebraic variety $\mathcal{R}[n,r,s]$ that contains all matrices with r-rigidity at most s. Our strategy to prove lower bounds for the rigidity of a matrix will be to prove non-membership in $\mathcal{R}[n,r,s]$.

We follow the discussion of [GHIL16], Sec. 2.1. Given n, r, s, define

$$\mathcal{R}^0[n,r,s] := \{A \in \mathrm{Mat}_n : \mathrm{Rig}_r(A) \leq s\},$$

the set of all matrices having r-rigidity at most s. We could use necessary conditions for membership in $\mathcal{R}^0[n,r,s]$ in order to prove lower bounds on the rigidity of a given matrix. In particular, we will be interested in polynomials $f \in \mathbb{C}[\mathrm{Mat}_n]$ that vanish identically on $\mathcal{R}^0[n,r,s]$: if we determine such a polynomial and a matrix $A \in \mathrm{Mat}_n$ such that $f(A) \neq 0$, we can deduce that $\mathrm{Rig}_r(A) > s$.

If f is a polynomial vanishing identically on $\mathcal{R}^0[n,r,s]$, then it vanishes identically on the Zariski (or equivalently Euclidean by Theorem 2.7) closure $\overline{\mathcal{R}^0[n,r,s]}$.

Definition 3.6. Define

$$\mathcal{R}[n,r,s] := \overline{\mathcal{R}^0[n,r,s]}.$$

We say that a matrix A has *r-border rigidity s* if s is the minimum integer such that $A \in \mathcal{R}[n,r,s]$; in this case we write $\underline{\mathrm{Rig}}_r(A) := s$.

Notice that $\mathrm{Rig}_r(A) \geq \underline{\mathrm{Rig}}_r(A)$. According to Definition 1.1, $A \in \mathcal{R}^0[n,r,s]$ if and only if $A = B+C$ where $B \in \sigma_r$ and C has at most s nonzero entries, namely $C \in L^S$ for some S consisting of s pairs of indices. We deduce

$$\mathcal{R}[n,r,s] = \bigcup_{|S|=s} \overline{\{A = B + C : B \in \sigma_r, C \in L^S\}},$$

or equivalently

$$\mathcal{R}[n,r,s] = \bigcup_{|S|=s} J(\sigma_r, L^S). \tag{3.2}$$

We have the following result

Proposition 3.7. *If $s < (n-r)^2$ then the affine variety $\mathcal{R}[n,r,s]$ is a proper subvariety of Mat_n with at most $\binom{n^2}{s}$ irreducible components. Moreover, it is equidimensional of dimension $r(2n-r)+s$.*

Proof. The joins $J(\sigma_r, L^S)$ of (3.2) are irreducible by Proposition 3.2. Moreover the value $\dim \sigma_r + \dim L^S = r(2n-r)+s$ is the expected dimension of $J(\sigma_r, L^S)$. We are going to show that if $J(\sigma_r, L^S)$ does not have the expected dimension then there exists an S^* with $|S^*| = s$ such that $J(\sigma_r, L^S) \subseteq J(\sigma_r, L^{S^*})$ and $J(\sigma_r, L^{S^*})$ has the expected dimension. In particular, this will show that only the joins of expected dimension are irreducible components of $\mathcal{R}[n,r,s]$.

First, we observe that if S' is such that $J(\sigma_r, L^{S'})$ does not coincide with the ambient space Mat_n, then there exists $S'' \supseteq S'$ such that $|S''| = |S'|+1$ and $\dim J(\sigma_r, L^{S''}) = \dim J(\sigma_r, L^{S'}) + 1$. If this was not the case, then, by the irreducibility of the joins, $J(\sigma_r, L^{S'}) = J(\sigma_r, L^{S''})$ for every $S'' \supseteq S'$ with $|S''| = |S'| + 1$, and recursively $J(\sigma_r, L^{S'}) = J(\sigma_r, L^{S^*})$ for every $S^* \supseteq S'$; since $J(\sigma_r, L^{S'}) \subsetneq \mathrm{Mat}_n$, we obtain a contradiction.

Now, if $\dim J(\sigma_r, S) = r(2n-r) + s'$ with $s' < s$ (and it is not the entire Mat_n), consider $S' \subseteq S$ with $|S'| = s'$ and $J(\sigma_r, L^{S'})$ has the expected dimension, so by irreducibility $\dim J(\sigma_r, S) = \dim J(\sigma_r, S')$. Now consider $S^* \supseteq S'$ with $|S^*| = s$ and $J(\sigma_r, S^*)$ has the expected dimension. Then $\dim J(\sigma_r, S) \subseteq \dim J(\sigma_r, S^*)$ and the latter is an irreducible component of $\mathcal{R}[n, r, s]$. $\square$

Remark 3.8 (A short detour on matroids)**.** The irreducible decomposition of $\mathcal{R}[n, r, s]$ provides a *matroid structure* on the power set of $\{1, \dots, n\} \times \{1, \dots, n\}$. We refer to [Oxl06] for generalities on Matroid Theory.

Fix n, r and define the *independent sets* of a matroid as follows:

$$\mathcal{I}_{n,r} = \{S \subseteq \{1, \dots, n\} \times \{1, \dots, n\} : J(\sigma_r, L^S) \text{ has the expected dimension}\}.$$

Bases of the matroid are given by

$$\mathcal{B}_{n,r} = \{S : |S| = (n-r)^2 \text{ and } J(\sigma_r, L^S) = \mathrm{Mat}_n\}.$$

The *rank function* is given by $S \mapsto \dim J(\sigma_r, L^S) - r(2n-r)$. In particular the *flats* are sets S such that if $S_1 \supseteq S$ then $J(\sigma_r, L^S) \subsetneq J(\sigma_r, L^{S_1})$.

A geometric version of Problem 1.3 can be phrased as follows:

Problem 3.9 (Valiant's Problem from a Geometric Perspective)**.** Find an explicit infinite family of matrices $A_n \in \mathrm{Mat}_n$ such that there exist $\varepsilon, \delta > 0$ with $A_n \notin \mathcal{R}[n, \varepsilon n, n^{1+\delta}]$.

A solution to Problem 3.9 would provide a solution to Problem 1.3 because $\mathrm{Rig}_r(A) >$

$\underline{\mathrm{Rig}}_r(A)$ for every A.

Proposition 3.7 shows that if A_n is a generic point of Mat_n, then $A_n \notin \mathcal{R}[n, r, (n - r)^2 - 1]$ for every n, r; in particular $\mathrm{Rig}_{\varepsilon n}(A_n) \geq \underline{\mathrm{Rig}}_{\varepsilon n}(A_n) = (1-\varepsilon)^2 n^2$.

Similarly, we can rephrase Problem 1.4 in terms of membership in an algebraic variety. Define

$$\mathcal{D}[n, r] := \bigcup_{\tau \in \mathfrak{S}_n} J(\sigma_r^{(n)}, L^{S^\tau})$$

where $S^\tau = \{(1, \tau(1)), \ldots, (n, \tau(n))\}$ so that L^{S^τ} is the linear space of matrices supported at the support of the permutation matrix corresponding to τ. The same argument that we used in Proposition 3.7 shows the following:

Proposition 3.10. *If $n \leq (n-r)^2 - 1$, then the affine variety $\mathcal{D}[n, r]$ has exactly $n!$ irreducible components. Moreover, it is equidimensional of dimension $r(2n - r) + n$.*

Proof. Notice all the $J(\sigma_r, L^{S^\tau})$, for $\tau \in \mathfrak{S}_n$, are isomorphic and distinct. In particular this proves that the number of irreducible components if $n!$ and the equidimensionality. Finally, if S has exactly one element in each row and each column, then $J(\sigma_r, L^S)$ has the expected dimension. This follows from the same argument as 3.7, and the fact that $J(\sigma_r, L^S)$ is invariant under simultaneous permutation of rows and columns. $\square$

We rephrase Problem 1.4 as follows

Problem 3.11 (Barak's Problem from a Geometric Perspective)**.** Find an explicit sequence of matrices $\{A_n\}$, with $A_n \in \mathrm{Mat}_n$ and $A_n \notin \mathcal{D}[n, \varepsilon n]$.

Every solution for Problem 3.9 is a solution for Problem 3.11

Notice that Problem 3.9 and Problem 3.11 are potentially *harder* than Problem 1.3

and Problem 1.4, because working with Zariski closed objects we will not be able to detect matrices of high rigidity if their border rigidity is low. However $\mathcal{R}[n, r, s]$ and $\mathcal{D}[n, r]$ are proper subvarieties of Mat_n in the range that we are interested in, therefore these problems can be still regarded as problems of finding hay in a haystack, as the original problems.

The next section studies the degrees of the irreducible components $J(\sigma_r, L^S)$.

3.3 Degree of cones over determinantal varieties

Bounds on the degree of $\mathcal{R}[n, r, s]$ were used in [Lok06] and [KLPS14] to construct matrices of high rigidity in Valiant's range (according to Problem 3.9). The construction provides indeed infinite sequences of rigid matrices in the range of Valiant's problem, but it does not satisfy the requirement of explicitness. More precisely, the sequence of [Lok06] uses roots of unity of order that grows exponentially in n; the construction of [KLPS14] makes use of roots of large primes: in both cases, the algebraic numbers that are used cannot be produced in polynomial time by a Turing machine, which makes the construction non-explicit.

This section deals with the degrees of the joins $J(\sigma_r, L^S)$, in the restricted range where $s \leq n$. We start with a classical result: it is a generalization of the classical Bezout's Theorem that will be useful in determining a recursive relation for the degrees of the joins that we are interested in. Only for this section, we work exclusively in projective space, using the same notation that we used before. Notice that all the varieties that we deal with are affine cones over projective varieties. Only for this section, it will be useful to work with the projective version of these varieties. We will not change the notation, so σ_r, L^S and $J(\sigma_r, L^S)$ are the projective varieties whose affine cones are the affine varieties that have been defined in the previous sections.

Proposition 3.12. *Let $X \subseteq \mathbb{P}V$ be a projective variety and let $x \in X$ such that $J(X,x) \neq X$. Define*

$$\pi_X : X \dashrightarrow \mathbb{P}(V/\widehat{x})$$

to be the restriction of the projection $\pi : \mathbb{P}V \dashrightarrow \mathbb{P}(V/\widehat{x})$. Then π_X is generically finite to 1 and

$$\deg J(X,x) = \frac{1}{\deg \pi_X}\left[\deg(X) - \deg TC_x X\right].$$

Proof. The condition $J(X,x) \neq X$ implies that π_X is finite to 1: if it was not, the fiber over $\pi_X(p)$ (for a generic $p \in X$) would contain the line $\langle x,p\rangle$ and by genericity we would have $\langle x,p\rangle \subseteq X$ for every $p \in X$, thus $J(X,x) = X$.

Moreover $J(X,x) \supsetneq X$ implies, by irreducibility, that $J(X,x)$ has the expected dimension, $\dim J(X,x) = \dim X + 1$ and $c = \operatorname{codim} J(X,x)$ (in $\mathbb{P}V$) is the same as $\operatorname{codim} \pi_X(X)$ (in $\mathbb{P}(V/x)$). From [Mum95], Thm. 5.11, we have

$$\deg \overline{\pi_X(X)} = \frac{1}{\deg \pi_X}\left[\deg(X) - \deg TC_x X\right].$$

Let $L = \mathbb{P}^c$ be a generic linear space not containing x, so that $J(X,x) \cap L$ contains exactly $\deg(J(X,x))$ points. Since $x \notin L$, the linear projection π maps it isomorphically to a subspace $\pi(L)$ of $\mathbb{P}(V/x)$ that intersects $\pi_X(X)$ in $\deg(\pi_X(X))$ points. Thus, we conclude that $\deg J(X,x) = \deg \pi_X(X)$. □

Fix $S = \{(i_1,j_1),\ldots,(i_s,j_s)\}$, and define $S_0 = \emptyset$, $S_k = S_{k-1} \cup \{(i_k,j_k)\}$. Clearly, for every k, $\widehat{L}^{S_k} = \widehat{L}^{S_{k-1}} + \widehat{L}^{(i_k,j_k)}$, therefore

$$J(\sigma_r, L^{S_k}) = J\left(J(\sigma_r, L^{S_{k-1}}), L^{\{(i_k,j_k)\}}\right). \tag{3.3}$$

In the case where S has not two entries in the same row or column, we will use the relation (3.3) and Proposition 3.12 recursively to determine the degree of $J(\sigma_r, L^S)$. Without loss of generality, we may assume $S = \{(1,1), \ldots, (s,s)\}$ is diagonal, and $s \le (n-r)^2$, otherwise $J(\sigma_r, L^S) = \mathrm{Mat}_n$.

First, we show that in this setting, the degree of the map π_X of Proposition 3.12 is 1.

Proposition 3.13. *Fix n, r, s with $s \le \min\{n, (n-r)^2\}$. Let $S = \{(1,1), \ldots, (s,s)\}$, $S' = S \smallsetminus \{(1,1)\}$ and let Z be the matrix with 1 in the top left entry and 0 elsewhere. Let $\pi : J(\sigma_r, L^{S'}) \to \mathbb{P}(\mathrm{Mat}_n/\langle Z\rangle)$ denote the projection from $[Z] \in \mathbb{P}Mat_n$. Then* $\deg \pi = 1$.

Proof. We want to prove that the generic fiber of π has 1 point. Equivalently, a generic line through $[Z]$ that intersects $J(\sigma_r, L^{S'})$, does it in a single point. By genericity, it suffices to find one point $[A] \in J(\sigma_r, L^{S'})$ such that the fiber $\pi^{-1}(\pi([A]))$ consists of the single point $\{[A]\}$; equivalently it suffices to find a matrix $A = B + C$ with $[B] \in \sigma_r, [C] \in L^{S'})$ with the property that does not exist $[B'] \in \sigma_r$, $[C'] \in L^{S'}$ such that $B + C = uZ + v(B' + C')$ for some $u, v \in \mathbb{C}$ unless $u = 0$, $B' = B$ and $C' = C$. This guarantees that the line through $[A]$ and $[Z]$ intersects $J(\sigma_r, \mathbb{P}L^{S'})$ only at $[A]$ and $[Z]$.

Let A be the matrix

$$A = \left(\begin{array}{c|c} 0 & \\ \hline & A_1 \end{array}\right)$$

where the blocking is $(1, n-1) \times (1, n-1)$, and $A_1 = B_1 + C_1$ is a generic point of $\widehat{J}(\sigma_r^{(n-1)}, L^{S'}) \subseteq \mathrm{Mat}_{n-1}$; let B, C be respectively B_1, C_1 extended with a row and a column of zeros. If $A = uZ + v(B' + C')$ with $u \neq 0$, $[B'] \in \sigma_r$ and $[C'] \in L^{S'}$, then the top left entry of B' has to be nonzero and the rest of the first row and column has to be 0. In particular $\mathrm{rank}(B') = 1 + \mathrm{rank}(B'_1) \leq r$, where B'_1 is the $(n-1)\times(n-1)$ bottom right block of B. This means that $A_1 = v(B'_1 + C'_1) \in \widehat{J}(\sigma_{r-1}^{(n-1)}, L^{S'})$; since $[A_1]$ is generic in $J(\sigma_r^{(n-1)}, L^{S'})$ this is possible only if $J(\sigma_{r-1}^{(n-1)}, L^{S'}) = \mathbb{P}\mathrm{Mat}_{n-1}$; but we have $\operatorname{codim} J(\sigma_{r-1}^{(n-1)}, L^{S'}) = (n-r)^2 - (s-1) > 0$ because $s \leq (n-r)^2$.

This shows that the fiber of π over $\pi([A])$ consists of exactly the point $[A]$, namely $\deg \pi = 1$. □

The following result was conjectural in an early version of [GHIL16]; it was then proved in a stronger form in [Alu15]. We propose here a version that is suitable to our setting, with an elementary proof.

Proposition 3.14. *Let $X \subseteq \mathbb{P}V$ and let $x \in X$. Let $p \in \mathbb{P}V$, $p \neq x$, with the property that $\pi : X \dashrightarrow \mathbb{P}(V/\widehat{p})$ is generically finite and* $\deg \pi = 1$. *If $TC_x J(X,p)$ is reduced and $TC_x X$ is not a cone over p, then $TC_x(J(X,p)) = J(TC_x X, p)$.*

Proof. First, observe that $TC_x(J(X,p))$ and $J(TC_x X, p)$ have the same dimension $\dim X + 1$. Since $J(X,p) \neq X$, we have that $J(X,p)$ has the expected dimension $\dim X + 1$, so $\dim TC_x(J(X,p)) = \dim X + 1$. On the other hand, $\dim TC_x X = \dim X$ and $J(TC_x X, p)$ has the expected dimension unless $TC_x X$ is a cone over p, but this would be in contradiction with the hypothesis.

We show the equality between $TC_x(J(X,p))$ and $J(TC_x X, p)$ by proving a double inclusion. Suppose $x = [x_0]$ for $x_0 \in \widehat{X}$ and $p = [p_0]$.

Let $z \in \widehat{J}(TC_x X, p)$, so $z = w + \alpha p_0$ for some $w \in TC_{x_0}\widehat{X}$ and $\alpha \in \mathbb{C}$. The vector w is tangent to $\widehat{X}$ at x_0, so $w = \frac{d}{dt}|_{t=0}(\omega(t))$, for a curve $\omega(t)$ in $\widehat{X}$ with $\omega(0) = x_0$. Define $\zeta(t) = \omega(t) + \alpha t p_0$. Then $\zeta(t)$ is a curve $\widehat{J}(X,p)$; since $\zeta(0) = x_0$ and $z = \frac{d}{dt}|_{t=0}\zeta(t)$, we conclude $[z] \in TC_{x_0}(J(X,p))$.

Conversely, suppose $z \in TC_{x_0}\widehat{J}(X,p)$. Then $z = \frac{d}{dt}|_{t=0}\zeta(t)$ for a curve $\zeta(t)$ in $\widehat{J}(X,p)$. By genericity assumption, we have $\zeta(t) = \xi(t) + \alpha(t)p_0$ for a curve $\xi(t) \in X$ and $\alpha(t) \in \mathbb{C}$; since $\deg \pi = 1$, ξ and α are uniquely determined by ζ, $\xi(0) = x_0$ and $\alpha(0) = 0$. We conclude $z = \frac{d}{dt}|_{t=0}\zeta(t) = \frac{d}{dt}|_{t=0}(\xi(t) + \alpha(t)p_0) = v + \alpha' p_0$ for some $v \in TC_{x_0}X$ and $\alpha' \in \mathbb{C}$, showing $z \in J(TC_x X, p)$. $\square$

By applying 3.14 recursively, we deduce

Corollary 3.15. *Let $X \subseteq \mathbb{P}V$ and let $x \in X$. Let $L \simeq \mathbb{P}^s \subseteq \mathbb{P}V$, $x \notin L$, with the property that $\pi : X \dashrightarrow \mathbb{P}(V/\widehat{L})$ is generically finite and $\deg \pi = 1$. If $TC_x J(X,L)$ is reduced and $TC_x X$ is not a cone over any point of L, then $TC_x(J(X,L)) = J(TC_x X, L)$.*

We recall this easy remark (see [Har92], Sec. 18.17, Calculation III):

Lemma 3.16. *Let $X \subseteq \mathbb{P}V$ and let $L = \mathbb{P}^m \subseteq \mathbb{P}V$ such that L is disjoint from the span of X. The $\pi : \mathbb{P}V \dashrightarrow \mathbb{P}(V/\widehat{L})$ is regular, finite and of degree 1 on X. Moreover $\deg X = \deg \pi(X) = \deg(J(X,L))$.*

Write $d_{n,r,s} := \deg J(\sigma_r^{(n)}, L^S)$, where S is the set of the first s diagonal entries, $S = \{(1,1), \dots, (s,s)\}$; set $d_{n,r,s} = 0$ if r is negative. In particular, $d_{n,r,0} = \deg \sigma_r^{(n)}$ is given by (3.1).

Proposition 3.17. *For every n, r, s with $s \leq \min\{(n-r)^2\}$, we have*

$$d_{n,r,s} = d_{n,r,s-1} - d_{n-1,r-1,s-1}.$$

Proof. Let p be the class in $\mathbb{P}Mat_n$ of the matrix with 1 at the (s,s) entry and 0 elsewhere. Let $S' = S \smallsetminus \{(s,s)\}$. We have $\widehat{L}^S = \widehat{L}^{S'} + \widehat{p}$. From 3.15, we have

$$\begin{aligned} TC_p J(\sigma_r^{(n)}, L^{S'}) = J(TC_p\sigma_r^{(n)}, L^{S'}) = \\ &= J(J(T_p\sigma_1^{(n)}, \sigma_{r-1}^{(n-1)}), L^{S'}) = \\ &= J(J(\sigma_{r-1}^{(n-1)}, L^{S'}), T_p\sigma_1^{(n)}). \end{aligned}$$

Here $\widehat{J}(\sigma_{r-1}^{(n-1)}, L^{S'})$ spans the subspace of matrices that are identically 0 in the s-th row and the s-th column. In particular the space $T_p\sigma_1^{(n)}$ is disjoint from the space spanned by $J(\sigma_{r-1}^{(n-1)}, L^{S'})$ in $\mathbb{P}Mat_n$. By Lemma 3.16, we have $\deg J(J(\sigma_{r-1}^{(n-1)}, L^{S'}), T_p\sigma_1^{(n)}) = \deg J(\sigma_{r-1}^{(n-1)}, L^{S'})$.

Observe $J(\sigma_r^{(n)}, L^S) = J(J(\sigma_r^{(n)}, L^{S'}), \{p\})$. Using Proposition 3.12, since $\deg \pi = 1$ by Proposition 3.13, we have

$$\begin{aligned} d_{n,r,s} &= \deg J(\sigma_r^{(n)}, L^S) = \\ &= \deg J(J(\sigma_r^{(n)}, L^{S'}), \{p\}) = \\ &= \deg J(\sigma_r^{(n)}, L^{S'}) - \deg TC_p J(\sigma_r^{(n)}, L^{S'}) = \\ &= \deg J(\sigma_r^{(n)}, L^{S'}) - \deg J(\sigma_{r-1}^{(n-1)}, L^{S'}) = d_{n,r,s-1} - d_{n-1,r-1,s-1}. \end{aligned}$$

□

We can use the recursive formula given by Proposition 3.17 to derive a formula for $\deg J(\sigma_r^{(n)}, L^S)$:

Theorem 3.18. *For every n, r, s with $s \leq (n-r)^2$*

$$d_{n,r,s} = \sum_{\ell=0}^{s} \binom{s}{\ell} (-1)^\ell d_{n-\ell, r-\ell, 0} \tag{3.4}$$

Proof. From the recursion $d_{n,r,s} = d_{n,r,s-1} - d_{n-1,r-1,s-1}$ of Proposition 3.17, we obtain by induction, for every $p \leq s$

$$d_{n,r,s} = \sum_{\ell=0}^{p} \binom{s}{\ell} (-1)^{\ell} d_{n-\ell,r-\ell,s-\ell}.$$

For $p = s$, we obtain (3.4). $\square$

In [Alu15], Paolo Aluffi investigates these degrees more thoroughly, using advanced techniques based on Fulton-MacPherson intersection theory (see [Ful84]). He studies the image $\overline{\pi_S(\sigma_r^{(n)})}$ via the projection $\pi_S : \mathbb{P}\mathrm{Mat}_n \dashrightarrow \mathbb{P}(\mathrm{Mat}_n/L^S)$ for several configurations S. To do this, he resolves the indeterminacy of π_S by blowing-up $\sigma_r^{(n)}$ along $\vartheta_{n,r,S} := \sigma_r^{(n)} \cap \mathbb{P}L^S$, and obtains a surjective, generically finite, regular map $Bl_{\vartheta_{n,r,S}}\sigma_r^{(n)} \to \overline{\pi_S(\sigma_r^{(n)})}$. He proves a generalization of Proposition 3.14 and reduces the problem of computing $\deg(Bl_{\vartheta_{n,r,S}})$ to a problem of *excess intersection*. The solution of the problem is obtained in terms of the *Segre class* $s(\vartheta_{n,r,S}, \sigma_r^{(n)})$ and a modified version of it obtained by a twisting operation described in [Alu94]. The computation of these characteristic classes is via the classical resolution of the singularities of $\sigma_r^{(n)}$ (see e.g. [ACGH85]).

This method leads to an alternative proof of the results of this section, and several generalizations. In particular, [Alu15] provides a recursive technique that applies when L^S has a block diagonal structure: this generalizes our method, where we consider a diagonal S, so the blocks are of size 1. With this method and some *base cases* that play as fundamental bricks of L^S, the calculation of $\deg(J(\sigma_r^{(n)}, L^S)$ for a several different configurations S follows.

4. EQUATIONS FOR CONES OVER DETERMINANTAL VARIETIES

In this chapter we study equations for $\mathcal{R}[n,r,s]$. Recall that our main goal is to find necessary conditions for a matrix to have low (border-)rigidity, in order to prove lower bounds on the rigidity of a given matrix A by showing that A does not satisfies these necessary conditions. Polynomial equations vanishing identically on $\mathcal{R}[n,r,s]$ are the conditions that we seek.

We saw that in general $\mathcal{R}[n,r,s] \subsetneq Mat_n$ is a reducible variety; to obtain an equation for a reducible variety, we can multiply an equation for each of its irreducible components. Therefore, our goal reduces to determining an equation for $J(\sigma_r, L^S)$ for every choice of S with $|S| = s$.

From Lemma 3.3, we have that $J(\sigma_r, L^S) = \pi_S^{-1}(\overline{\pi_S(\sigma_r)})$, where $\pi_S : V \to V/L^S$ is the canonical projection. The pull back map

$$\pi_S^* : \mathbb{C}[V/L^S] \to \mathbb{C}[V]$$

is, in coordinates, the ring embedding

$$\mathbb{C}[x^i_j : (i,j) \notin S] \to \mathbb{C}[x^i_j : i,j = 1,\ldots,n].$$

By Lemma 2.5, we observe that equations for $J(\sigma_r, L^S)$ can be obtained from equations of $\overline{\pi_S(\sigma_r)}$. There is a rich theory that develops methods to produce equations for (the closures of) a projection of an algebraic variety: it is called *elimination theory.* Elimination theory has been studied in many forms in commutative algebra, algebraic geometry and invariant theory; it has connection with logic, as the *quantifier elimination* can be regarded as an instance of elimination theory: indeed, quantified formulas in first order logic are equivalent to statements about membership in the image of a projection.

The following result is of great importance in our strategy to determine equations for $J(\sigma_r, L^S)$.

Theorem 4.1 (Fundamental Theorem of Elimination Theory, see e.g. [CLO07], Thm. 3.2.3)**.** *Let $X \subseteq V$ be an affine variety with ideal $I \subseteq \mathbb{C}[V]$. Let L be a linear subspace of V and let $\pi : V \to V/L$ be the projection. Then $I(\overline{\pi(X)}) = I \cap \mathbb{C}[V/L]$ (where $\mathbb{C}[V/L]$ is viewed as a subring of $\mathbb{C}[V]$). In coordinates, if $x_1, \ldots, x_n$ are coordinates on V and $L = \{x_1 = \cdots = x_\ell = 0\}$, then $I(\overline{\pi(X)}) = I \cap \mathbb{C}[x_{\ell+1} \cdots x_n]$.*

In general, elimination theory can be performed using Groebner bases techniques, which makes the calculation of elimination ideals computationally hard. We refer to [CLO07] for the theory of Groebner bases. Here, we only state the main result that allows us to compute elimination ideals.

Theorem 4.2 (see e.g. [CLO07], Thm. 3.1.2)**.** *Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal and let G be a Groebner basis of I with respect to the monomial order* Lex *with $x_1 > \cdots > x_n$. Then, for every $\ell = 1, \ldots, n$*

$$G_\ell = G \cap \mathbb{C}[x_1, \ldots, x_\ell]$$

is a Groebner bases for $I \cap \mathbb{C}[x_1, \ldots, x_\ell]$.

Theorem 4.2 is the main tool used by software packages that perform elimination theory, for instance Macaulay2 (see [GS]). In [GHIL16], we computed the ideal of

$J(\sigma_r, L^S)$ in several cases for small values of n and r (especially in the case where $J(\sigma_r, L^S)$ is a hypersurface).

From Theorem 4.1, we deduce that equations for $\overline{\pi_S(\sigma_r)}$ (and so for $J(\sigma_r, L^S)$) are (generated by) polynomials in $I(\sigma_r) \cap \mathbb{C}[\mathrm{Mat}_n/L^S] = I(\sigma_r) \cap \mathbb{C}[x^i_j : (i,j) \notin S]$. In particular, f is an equation for $J(\sigma_r, L^S)$ if the following hold:

- f is generated by minors of size $r+1$;
- f does not involve entries in S.

A particular case occurs when there exists a minor of size $r+1$ that does not involve entries in S; in this case these minors provide equations of degree $r+1$ for $J(\sigma_r, L^S)$, that we call *avoiding minors*. We will see that avoiding minors alone are enough to solve Problem 3.11 in the trivial case $\varepsilon < 1/2$. However, the presence of avoiding minors is only possible if either r or s are small or if S has a particular structure; in most cases, this is not what happens.

In this chapter, we present some explicit methods to determine equations for $J(\sigma_r, L^S)$. Since $J(\sigma_r, L^S)$ is a component of $\mathcal{R}[n, r, s]$ if and only if it has the expected dimension, we can restrict to this case. Moreover, since we are only interested in determining a single equation for every $J(\sigma_r, L^S)$, it is not restrictive to consider only the case where $J(\sigma_r, L^S)$ is a hypersurface since for every S there exists S^* such that $J(\sigma_r, L^S) \subseteq J(\sigma_r, L^{S^*})$ and the latter is a hypersurface of the expected dimension. Ideally, one would like to work with the *flats* of the matroid defined in Remark 3.8, but these seem hard to describe in general.

If $J(\sigma_r, L^S)$ is a hypersurface of the expected dimension, then $s = (n-r)^2 - 1$.

The following result is useful to rule out configurations S that do not give to a join of expected dimension

Lemma 4.3 ([GHIL16], Lemma 3.4)**.** *Let* $S' = \{(1,1), \dots, (n-r,1)\}$ *and* $S = S' \cup \{((n-r+1,1), \dots, (n,1)\}$*. Then* $J(\sigma_r, L^{S'}) = J(\sigma_r, L^S)$*.*

Proof. Since $S' \subseteq S$, $J(\sigma_r, L^{S'}) \subseteq J(\sigma_r, L^S)$.

To prove the other inclusion, consider a generic $A \in J(\sigma_r, L^S)$; the submatrix of A obtained removing the first column has rank r. Let $\widetilde{A}$ be the $r \times r$ bottom right submatrix of A: by genericity $\widetilde{A}$ is non singular, therefore there are unique $c_{n-r+1}, \dots, c_n$ such that

$$\begin{pmatrix} a_1^{n-r+1} \\ \vdots \\ a_1^n \end{pmatrix} = \sum_{j=n-r+1}^{n} c_j \begin{pmatrix} a_j^{n-r+1} \\ \vdots \\ a_j^n \end{pmatrix}.$$

Let B be the matrix defined by $b_j^i = a_j^i$ if $j \geq 2$

$$\begin{pmatrix} b_1^1 \\ \vdots \\ b_1^n \end{pmatrix} = \sum_{j=n-r+1}^{n} c_j \begin{pmatrix} a_j^1 \\ \vdots \\ a_j^n \end{pmatrix}.$$

Then $B \in \sigma_r$ and $A = B + C$ with $C \in L^{S'}$, so $A \in J(\sigma_r, L^{S'})$. $\square$

Lemma 4.3 implies that if S contains more than $n-r$ entries in a single row or column, then $J(\sigma_r, L^S)$ does not have the expected dimension unless it is the entire space Mat_n.

In Section 4.1, we present the results of Section 3.2 and 3.4 of [GHIL16]. In Section 4.2, we develop a new method to produce equations in the case where S has only one element in each row and column: in particular, this can be an approach to attack Problem 3.11. In section 4.3, we study the new method that we introduce from a representation theoretic point of view.

4.1 Extreme cases: $r = 1$ and $r = n - 2$

In this section, we completely characterize the components of $\mathcal{R}[n, r, s]$ in the cases $r = 1$ and $r = n - 2$, in the hypersurface case.

Case $r = 1$

If $r = 1$, then $\dim \sigma_1^{(n)} = 2n - 1$, that is $\operatorname{codim} \sigma_1^{(n)} = n^2 - 2n + 1$, so that $\mathcal{R}[n, 1, n^2 - 2n + 1]$ is a hypersurface. We will analyze $J(\sigma_1, L^S)$ where $|S| \geq n^2 - 2n + 1$. It will be convenient to describe the configurations S by describing their complement configurations S^c.

Proposition 4.4. *Let* $2 \leq k \leq n$ *and* $S^c \supseteq \{(1,1), \ldots, (k,k), (1,2), \ldots, (k-1,k), (k,1)\}$. *Then*

$$x_1^1 \cdots x_k^k - x_2^1 \cdots x_k^{k-1} x_1^k \tag{4.1}$$

is an equation for $J(\sigma_1, L^S)$. *Moreover, if* $S^c = \{(1,1), \ldots, (k,k), (1,2), \ldots, (k-1,k), (k,1)\}$, *then* $J(\sigma_1, L^S)$ *is a hypersurface and* (4.1) *is its (unique up to scale) equation.*

Proof. Let $f = x_1^1 \cdots x_k^k - x_2^1 \cdots x_k^{k-1} x_1^k$. We will show that $f \in I(\sigma_1)$, namely that f is generated by 2×2 minors. Since f does not involve entries in S, we will conclude

$f \in I(J(\sigma_1, L^S))$. If $k = 2$, then $f = M_{12}^{12}$. Define recursively

$$f_2 := M_{12}^{12} = x_1^1 x_2^2 - x_2^1 x_1^2,$$
$$f_j := x_j^j f_{j-1} - x_2^1 \cdots x_{j-1}^{j-2} M_{1,j}^{j-1,j}.$$

By induction, we have $f_j = x_1^1 \cdots x_j^j - x_2^1 \cdots x_1^j \in I(\sigma_1)$ for all $j = 3, \ldots, k$ and $f_k = f$.

The last assertion follows from Lemma 4.3 and the discussion in the proof of Proposition 3.7: indeed, for any $S' \supsetneq S$, iterated applications of Lemma 4.3 implies $J(\sigma_1, L^{S'}) = \mathrm{Mat}_n$; if $J(\sigma_1, L^S)$ had codimension greater than 1, then there would be a $S' \supseteq S$ such that $J(\sigma_1, L^{S'})$ was a hypersurface, providing a contradiction. $\square$

It turns out that, up to permutations of rows and columns, Proposition 4.4 describes every hypersurface of the form $J(\sigma_1, L^S)$ as we will see in Theorem 4.7.

Lemma 4.5. *Let S be a configuration omitting at least two entries in each row and in each column. Then there exists $k \geq 2$ such that, up to a permutation of rows and columns, $S^c \supseteq \{(1,1), \ldots, (k,k), (1,2), \ldots, (k,1)\}$.*

Proof. After a permutation, we may assume $(1,1) \in S^c$, and, since S omits at least another entry in the first column, $(2,1) \in S^c$. Since S omits at least 2 entries in the second row, assume $(2,2) \in S^c$. S omits at least one more entry in the second column: if that entry is $(1,2)$, then $k = 2$ and S omits a 2×2 minor; otherwise we may assume $(3,2) \in S^c$. Again S omits another entry on the third row: if that entry is $(3,1)$ (resp. $(3,2)$), then $k = 3$ (resp. $k = 2$) and S omits a set of the desired form. After at most $2n$ steps, this procedure terminates, and we obtain that S^c contains a subset of entries in a $k \times k$ submatrix K with the following configurations, as desired:

$$\begin{bmatrix} \Diamond & \Diamond & & \\ & \ddots & \ddots & \\ & & \ddots & \Diamond \\ \Diamond & & & \Diamond \end{bmatrix}.$$

□

Lemma 4.6. *Let S be a configuration of $n^2 - 2n$ entries. Then there exist $k \in \{1, \dots, n\}$ and a $k \times k$ submatrix K such that, up to a permutation of rows and columns, at least $2k$ entries of the complement S^c of S lie in K in the following configuration*

$$\begin{bmatrix} \Diamond & \Diamond & & \\ & \ddots & \ddots & \\ & & \ddots & \Diamond \\ \Diamond & & & \Diamond \end{bmatrix}. \tag{4.2}$$

Moreover, if $J(\sigma_1, L^S)$ is a hypersurface then these are the only omitted entries in K and the ideal of $J(\sigma_1, L^S)$ is generated by $\left(x_1^1 \cdots x_k^k - x_2^1 \cdots x_1^k\right)$.

Proof. To prove the first assertion, we proceed by induction on n. The case $n = 2$ provides $s = 0$ and $k = 2$ trivially satisfies the statement.

First, suppose that S contains an entire row (or an entire column). Then S^c is concentrated in a $(n-1) \times n$ submatrix. In this case we may consider an $(n-1) \times (n-1)$ submatrix obtained by removing a column that contains at most 2 entries of S^c. Thus, up to reduction to a smaller matrix, we may always assume that S omits at least one entry in every row (and at least one entry in every column).

If S omits at least (and therefore exactly) 2 entries in each row and in each column, then we conclude by Lemma 4.5. So assume there is at least one row that omits only one entry.

After a permutation, we may assume $(1,1) \in S^c$ is the only entry in S^c in the first row. If the first column omits at most one more entry, then S omits at least $2(n-1)$ entries in the the submatrix obtained by removing the first row and the first column. We conclude by induction that, in this submatrix, there exists a k and a $k \times k$ submatrix with the desired configuration in the submatrix.

Finally, if the first column omits at least 2 entries other than x^1_1, then there is another column omitting only 1 entry. Consider the submatrix obtained by removing this column and the first row: S omits exactly $2(n-1)$ entries in this submatrix, and again we conclude by induction.

To prove the last assertion, if other omitted entries lie in K, then by Lemma 4.5 they provide another equation for $J(\sigma_1, L^S)$. □

Theorem 4.7 ([GHIL16], Thm 3.8)**.** *The variety $\mathcal{R}[n,1,n^2-2n]$ is a reducible hypersurface and the number of its irreducible components coincides with the number of cycles of the complete bipartite graph $K_{n,n}$, that is*

$$\sum_{k=2}^{n} \binom{n}{k}^2 \frac{k!(k-1)!}{2}.$$

Moreover, every ideal of an irreducible component is generated by a binomial of the form

$$x^{i_1}_{j_1} \cdots x^{i_k}_{j_k} - x^{i_1}_{j_{\tau(1)}} \cdots x^{i_k}_{j_{\tau(k)}},$$

for some k, where $\tau \in \mathfrak{S}_k$ is a cycle of length k and $I, J \subset\subseteq 1, \ldots, n$ have k elements *each.*

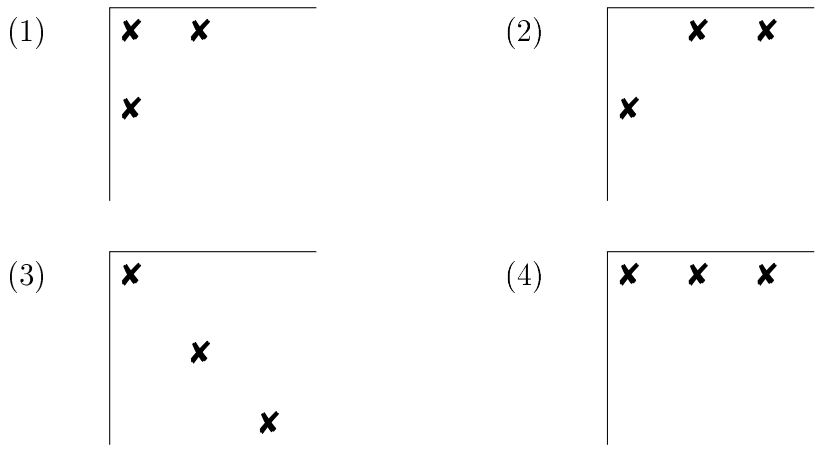
Proof. $\mathcal{R}[n,1,n^2-2n]$ is equidimensional and its irreducible components are $J(\sigma_1, L^S)$ where S is a configuration of n^2-2n entries providing a join of codimension 1.

Let S be a configuration of n^2-2n entries such that the corresponding join is a hypersurface. By Lemma 4.6 there exists a k such that, up to a permutation of rows and columns, S omits the entries $x_1^1, \dots, x_k^k, x_2^1, \dots, x_1^k$ and the equation of $J(\sigma_1, L^S)$ is $x_1^1 \cdots x_k^k - x_2^1 \cdots x_1^k = 0$. In particular, entries in S^c that do not lie in the submatrix K of Lemma 4.2 are free to vary. Let S^* be the set of entries whose complement is $\{x_1^1, \dots, x_k^k, x_2^1, \dots, x_1^k\}$; we obtain $J(\sigma_1, L^S) = J(\sigma_1, L^{S^*})$. This shows that the irreducible components are determined by the choice of a $k \times k$ submatrix and by the choice, in this submatrix, of a configuration of $2k$ entries such that, after a permutation of rows and columns, it has the form of (4.2).

Every configuration of this type, viewed as the adjacency matrix of a (n,n)-bipartite graph, determines a subgraph of the complete bipartite graph $K_{n,n}$ that is a cycle. This shows that the number of irreducible components of $\mathcal{R}[n,1,n^2-2n]$ is the number of such cycles. $\square$

Case r = n-2

We have $\operatorname{codim} \sigma_{n-2}^{(n)} = 4$, therefore $\mathcal{R}[n,n-2,3]$ is a hypersurface. We analyze joins $J(\sigma_{n-2}, L^S)$ with $|S| = 3$. Up to permutations of rows and columns and up to transpose, the entries in S are concentrated in the top left 3×3 submatrix, in one of the following forms



(4.3)

We will deal with each of the four cases. If i is an index, we denote by i^c the complement of $\{i\}$ in $\{1,\dots,n\}$. In configurations (1) and (2), the minor $M^{1^c}_{1^c}$ is an avoiding minor, because it does not involve entries in S, therefore it is the equation of $J(\sigma_{n-2}, L^S)$; in configuration (3), consider the two minors $M^{12(123)^c}_{13(123)^c}$ and $M^{13(123)^c}_{12(123)^c}$; they have size $n-1$ and the only entry of S that they involve is $(1,1)$. The polynomial

$$M^{3(123)^c}_{2(123)^c} M^{12(123)^c}_{13(123)^c} - M^{2(123)^c}_{3(123)^c} M^{13(123)^c}_{12(123)^c} \tag{4.4}$$

is irreducible of degree $2n-3$ and does not involve x^1_1; therefore it is the equation for $J(\sigma_{n-2}, L^S)$. Finally, configuration (4) does not provide a hypersurface, because of Lemma 4.3.

We conclude

Theorem 4.8 ([GHIL16], Thm 3.14)**.** *The variety $\mathcal{R}[n, n-2, 3]$ is a hypersurface and its irreducible components $J(\sigma_{n-2}, L^S)$ are of two types up to isomorphism:*

- *If S corresponds to type (1) or (2) of* (4.3)*, then* $\deg J(\sigma_{n-2}, L^S)$ *has degree $n-1$. There are n^2 such components.*

· *If S corresponds to type (3) of* (4.3)*, then* $\deg J(\sigma_{n-2}, L^S)$ *has degree* $2n-3$*. There are* $6\binom{n}{3}$ *such components.*

Proof. From the discussion above, we have that there are only two types of components and the degrees are the ones claimed in the statement. The number of components of the first type is n^2, as many as the minors of size $n-1$. The number of components of the second type is $6\binom{n}{3}^2$, as may as the choices of a 3×3 submatrix and three entries in it not sharing a row or a column. □

We conclude this section with an elementary observation on the equation corresponding to type (3) in (4.3). $M^{3(123)^c}_{2(123)^c}$ is the derivative in x^1_1 of $M^{13(123)^c}_{12(123)^c}$ and similarly $M^{2(123)^c}_{3(123)^c}$ is the derivative of $M^{12(123)^c}_{13(123)^c}$. We can rewrite (4.4) as

$$\det\begin{pmatrix} M^{13(123)^c}_{12(123)^c} & M^{12(123)^c}_{13(123)^c} \\ M^{3(123)^c}_{2(123)^c} & M^{2(123)^c}_{3(123)^c} \end{pmatrix} \tag{4.5}$$

or equivalently

$$\det\begin{pmatrix} M^{13(123)^c}_{12(123)^c} & M^{12(123)^c}_{13(123)^c} \\ \frac{\partial}{\partial x^1_1} M^{13(123)^c}_{12(123)^c} & \frac{\partial}{\partial x^1_1} M^{12(123)^c}_{13(123)^c} \end{pmatrix}.$$

In the next section, we will see that a generalization of this construction will provide a method to find equations for $J(\sigma_r, L^S)$, when S is diagonal.

4.2 Equations for $J(\sigma_r, L^S)$ via iterated determinants

In this section, we propose a more general approach that uses recursively the idea used to obtain (4.4), in order to determine equations for $J(\sigma_r, L^S)$ when S has exactly one entry in each row and each column. This is the case of Problems 1.4 and its

Zariski closed version (Problem 3.11). Without loss of generality, we will consider the diagonal case $S = \{(1,1), \dots, (n,n)\}$.

So, fix n and let $S = \{(1,1), \dots, (n,n)\}$ be the entire diagonal. Fix $r \leq n - \sqrt{n}$, so that $J(\sigma_r, L^S) \subsetneq \mathrm{Mat}_n$ is a proper subvariety. Let $\omega = 2(r+1) - n$. Notice that ω is the minimum number of diagonal entries that a minor of size $r+1$ has to involve. In particular, if $\omega \leq 0$, then there are avoiding minors of size $r+1$ (for instance the top right minor of size $r+1$). These minors provide equations for $J(\sigma_r, L^S)$. We will develop a method to obtain equations when $\omega \geq 1$.

In order to gain some intuition, before presenting the general construction, we present it in the particular cases where $\omega = 1$ and $\omega = 2$.

Case $\omega = 1$

If we have $n = 2r+1$, then $\omega = 2(r+1) - n = 1$. In this case, we can obtain an equation for $J(\sigma_r, L^S)$ using the same method that we used in (4.5). Figure 4.1 represents pictorially how the equations arise. We fix an index $c \in \{1, \dots, s\}$ (the central index $c = r+1$ in the picture), and two pairs of complementary sets of r indices (I_1, J_1) and (I_2, J_2) with $I_1 \cap J_1 = I_2 \cap J_2 = \emptyset$, so that $I_1 \cup J_1 = I_2 \cup J_2 = \{1, \dots, n\} \smallsetminus \{c\}$ (in the picture $I_2 = J_1 = \{1, \dots, r\}$ and $I_1 = J_2 = \{r+2, \dots, 2r-1\}$.

Notice that the minors $M^{I_1}_{J_1}$ and $M^{I_2}_{J_2}$ (the determinants of the submatrices boxed in dashed lines), do not involve diagonal entries. Therefore, the only diagonal entry in M^{c,I_1}_{c,J_1} and M^{c,I_2}_{c,J_2} (the determinants of the submatrices boxed in continuous lines) is x^c_c, that appears in degree at most 1 in every monomial in each of the two minors; moreover $M^{c,I_1}_{c,J_1} = x^c_c M^{I_1}_{J_1} + h_1$ where h_1 is a polynomial not involving x^c_c (nor other diagonal entries) and similarly M^{c,I_2}_{c,J_2}.

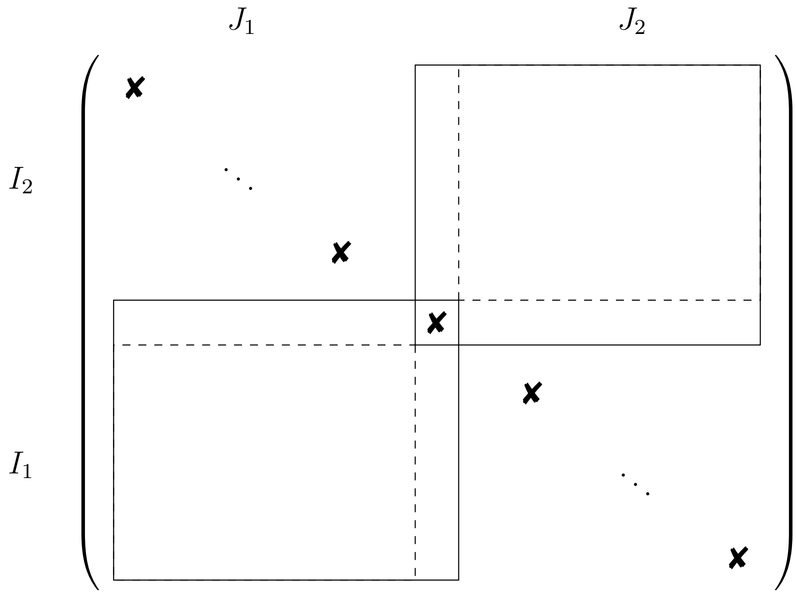


Figure 4.1: An equation when $\omega = 1$

The polynomial

$$M^{I_2}_{J_2} M^{c,I_1}_{c,J_1} - M^{I_1}_{J_1} M^{c,I_2}_{c,J_2}$$

does not involve the entry x^c_c, so it is an equation for $J(\sigma_r, L^S)$. We can rewrite this equation as

$$\det \begin{pmatrix} M^{c,I_1}_{c,J_1} & M^{c,I_2}_{c,J_2} \\ M^{I_1}_{J_1} & M^{I_2}_{J_2} \end{pmatrix} \tag{4.6}$$

and notice that the second row of the matrix is the derivative in x^c_c of the first row. More generally, we consider all minors of size $r+1$ of the form M^{cI}_{cJ} with $I \cap J = \emptyset$ and define a map that performs the operation (4.6) on pairs of them. Let

$$\mathbf{E}^{(n,r)}_1 = \left\langle M^{cI}_{cJ} : |I| = |J| = r, I \cap J = \emptyset \right\rangle,$$

that is the space of polynomials generated by minors of size $r+1$ involving the diagonal entry x_c^c and no other diagonal entries.

Define

$$\begin{aligned} \mathbf{E}_1^{(n,r)} \times \mathbf{E}_1^{(n,r)} &\to S^{2r+1}\mathrm{Mat}_n^* \\ (f_1, f_2) &\mapsto \det \begin{pmatrix} f_1 & f_2 \\ \frac{\partial}{\partial x_c^c} f_1 & \frac{\partial}{\partial x_c^c} f_2 \end{pmatrix}; \end{aligned}$$

the image of this map is a space of equations for $J(\sigma_r, L^S)$.

Since we prefer to work with linear maps, we will consider the induced map on $\mathbf{E}_1^{(n,r)} \otimes \mathbf{E}_1^{(n,r)}$; moreover, since the determinant is a skew-symmetric function, we can observe that we obtain the same image by restricting to the subspace $\bigwedge^2 \mathbf{E}_1^{(n,r)} \subseteq \mathbf{E}_1^{(n,r)} \otimes \mathbf{E}_1^{(n,r)}$.

Case $\omega = 2$

If $n = 2r$, then $\omega = 2(r+1) - n = 2$. In this case, we fix two indices c_1, c_2; without loss of generality, suppose they are 1 and 2. We will consider all minors of size $(r+1)$ of the form M_{12J}^{12I}, with $I \cap J = \emptyset$, so that $I \cup J = \{3, \dots, n\}$. In particular $M_{12J}^{12I} = x_1^1 x_2^2 M_J^I + h$, where h is a polynomial that is (affine) linear in x_1^1 and x_2^2 and does not involve other diagonal entries.

We apply the same technique we used before to a pair of minors of this form, in order to eliminate the term that is quadratic in $x_1^1 x_2^2$. Given $M_{12J_1}^{12I_1}$ and $M_{12J_2}^{12I_2}$ as above, consider the polynomial

$$g = \det \begin{pmatrix} M^{12I_1}_{12J_1} & M^{12I_2}_{12J_2} \\ M^{I_1}_{J_1} & M^{I_2}_{J_2} \end{pmatrix} \tag{4.7}$$

and notice that the second row of the matrix is the second derivative in x^1_1, x^2_2 of the first row. The polynomial g is (affine) linear in the variables x^1_1, x^2_2.

From a choice of 6 pairs (I, J), we can obtain three such polynomials g_1, g_2, g_3. At this point, we can consider the following 3×3 determinant:

$$f = \det \begin{pmatrix} g_1 & g_2 & g_3 \\ \partial_1 g_1 & \partial_1 g_2 & \partial_1 g_3 \\ \partial_2 g_1 & \partial_2 g_2 & \partial_2 g_3 \end{pmatrix}, \tag{4.8}$$

where ∂_1 and ∂_2 denote respectively the derivative in x^1_1 and x^2_2.

It is easy to show that f does not involve x^1_1 and x^2_2 (nor other diagonal variables). In particular, with this method we obtain equations for $J(\sigma_r, L^S)$ when $\omega = 2$.

As we did in the case $\omega = 1$, it is useful to define a linear map that performs the operations of (4.7) and (4.8). Let

$$\mathbf{E}^{(n,r)}_2 = \left\langle M^{12I}_{12J} : |I| = |J| = r - 1, I \cap J = \emptyset \right\rangle$$

that is the space of polynomials generated by minors of size $r + 1$ involving the diagonal entries x^1_1 and x^2_2 and no other diagonal entries.

We define two linear maps as follows. The first map produces polynomials that are affine linear in x_1^1, x_2^2

$$\begin{aligned} \bigwedge^2 \mathbf{E}_2^{(n,r)} &\to \mathbf{E}_1^{(n,r)} \subseteq S^{2r}\mathrm{Mat}_n^* \\ f_1 \wedge f_2 &\mapsto \det \begin{pmatrix} f_1 & f_2 \\ \partial_1\partial_2 f_1 & \partial_1\partial_2 f_2 \end{pmatrix}; \end{aligned}$$

here $\mathbf{E}_1^{(n,r)}$ denotes the image of the map in $S^{2r}\mathrm{Mat}_n^*$ (notice that the degree $2r$ arises as $(r+1)+(r-1)$). The second map uses the image of the first map to produce polynomials not involving diagonal entries:

$$\begin{aligned} \bigwedge^3 \mathbf{E}_1^{(n,r)} &\to \mathbf{E}_0^{(n,r)} \subseteq S^{6r-2}\mathrm{Mat}_n^* \\ g_1 \wedge g_2 \wedge g_3 &\mapsto \det \begin{pmatrix} g_1 & g_2 & g_3 \\ \partial_1 g_1 & \partial_1 g_2 & \partial_1 g_3 \\ \partial_2 g_1 & \partial_2 g_2 & \partial_2 g_3 \end{pmatrix}; \end{aligned}$$

here the degree arises as $2r + (2r-1) + (2r-1) = 6r-2$; the image of this second map is a space of equations for $J(\sigma_r, L^S)$.

General construction

Define

$$\mathbf{E}_\omega^{(n,r)} := \left\langle M_{1,\dots,\omega,J'}^{1,\dots,\omega,I'} : |I'| = |J'| = r+1-\omega, I' \cap J' = \emptyset \right\rangle \subseteq S^{r+1}(\mathrm{Mat}_n)^*. \tag{4.9}$$

The space $\mathbf{E}_\omega^{(n,r)}$ is the vector space generated by all minors of size $r+1$ involving no diagonal entries except $x_1^1, \dots, x_\omega^\omega$.

If $\tau = \{t_1, \dots, t_{|\tau|}\}$ is a set of indices, write $\partial_\tau := \dfrac{\partial^{|\tau|}}{\partial x_{t_1}^{t_1} \cdots \partial x_{t_{|\tau|}}^{t_{|\tau|}}}$; if $\tau = \emptyset$, then $\partial_\tau := 1$.

For every $\ell = \omega, \omega - 1, \dots, 1$, let $\Omega_\ell := \{\tau \subseteq \{1, \dots, \omega\} : |\tau| = \ell\}$, the set of subsets of $\{1, \dots, \omega\}$ with cardinality ℓ; let $\Omega_\ell^+ := \Omega_\ell \cup \{\emptyset\}$, so that $|\Omega_\ell^+| = 1 + \binom{n}{\ell}$. Let $d_\omega := r + 1$ and for every $\ell = \omega, \dots, 1$ define $d_{\ell-1} := \binom{n}{\ell}(d_\ell - \ell) + d_\ell$.

For every $\ell = \omega, \dots, 1$, we recursively define a map $\psi_\ell^{(n,r)}$ as follows

$$\begin{aligned} \psi_\ell^{(n,r)} : \textstyle\bigwedge^{1+\binom{\omega}{\ell}} \mathbf{E}_\ell^{(n,r)} &\to S^{d_{\ell-1}}(\mathrm{Mat}_n)^* \\ \wedge_{\rho \in \Omega_\ell^+} f_\rho &\mapsto \det(\partial_\tau f_\rho)_{\tau, \rho \in \Omega_\ell^+}. \end{aligned} \tag{4.10}$$

Set $\mathbf{E}_{\ell-1}^{(n,r)} := \mathrm{Im}\ \psi_\ell^{(n,r)}$. We will drop the superscript in $\psi_\ell^{(n,r)}$ if it does not create confusion. Every ψ_ℓ associates to the wedge product of $1 + \binom{\omega}{\ell}$ polynomials the determinant of a matrix size $1 + \binom{\omega}{\ell}$ whose rows are given by derivatives of order ℓ of the polynomials.

Proposition 4.9. *The elements of $\mathbf{E}_\ell^{(n,r)}$ have degree at most ℓ in the diagonal entries.*

Proof. First, notice that the variables $x_{\omega+1}^{\omega+1}, \dots, \omega_n^n$ do not appear in $\mathbf{E}_\ell^{(n,r)}$ for any ℓ.

We proceed by reverse induction on ℓ. For $\ell = \omega$, $\mathbf{E}_\omega^{(n,r)}$ is generated by the minors of the form $M_{1,\dots,\omega,J'}^{1,\dots,\omega,I'}$ with $I' \cap J' = \emptyset$. In particular these minors have degree ω in the diagonal entries: the terms of highest degree in these variables are given by $x_1^1 \cdots x_\omega^\omega M_{J'}^{I'}$.

For every $\rho \in \Omega_\ell^+$ consider $f_\rho \in \mathbf{E}_\ell^{(n,r)}$. By inductive hypothesis every f_ρ has degree at most ℓ in the diagonal entries, so if $\tau \in \Omega_\ell^+$ and $\tau \neq \emptyset$, we obtain that $\partial_\tau f_\rho$ does not depend on the diagonal entries.

Write $\rho_0, \dots, \rho_{N_\ell}$ for the elements of Ω_ℓ^+, with $\rho_0 = \emptyset$. The image of $\wedge_{\rho \in \Omega_\ell^+} f_\rho$ is (up to sign) the determinant of the following matrix of size $N_\ell + 1$:

$$F_\ell := \begin{pmatrix} f_{\rho_0} & \cdots & f_{\rho_{N_\ell}} \\ \partial_{\rho_1} f_{\rho_0} & \cdots & \partial_{\rho_1} f_{\rho_0} \\ \vdots & & \vdots \\ \partial_{\rho_{N_\ell}} f_{\rho_0} & \cdots & \partial_{\rho_{N_\ell}} f_{\rho_0} \end{pmatrix}. \tag{4.11}$$

Our goal is to show that the determinant of F_ℓ is of degree at most $\ell - 1$ in the diagonal entries, namely that if $\tau \in \Omega_\ell^+$, then $\partial_\tau \det(F_\ell) = 0$.

By inductive hypothesis, the diagonal entries $x_1^1, \dots, x_\omega^\omega$ only appear in the first row of F_ℓ. Therefore

$$\partial_\tau \det(F_\ell) = \det \begin{pmatrix} \partial_\tau f_{\rho_0} & \cdots & \partial_\tau f_{\rho_{N_\ell}} \\ \partial_{\rho_1} f_{\rho_0} & \cdots & \partial_{\rho_1} f_{\rho_0} \\ \vdots & & \vdots \\ \partial_{\rho_{N_\ell}} f_{\rho_0} & \cdots & \partial_{\rho_{N_\ell}} f_{\rho_0} \end{pmatrix}$$

that is 0 because $\tau = \rho_p$ for some p, so the matrix has two rows that are equal. $\square$

Finally, we can prove that this method provides equations testing for non-membership in $J(\sigma_r, L^S)$ when $S = \{(1,1), \dots, (n,n)\}$.

Theorem 4.10. $\mathbf{E}_0^{(n,r)}$ *is a space of equations for* $J(\sigma_r, L^S)$.

Proof. From Proposition 4.9, we obtain that the elements of $\mathbf{E}_0^{(n,r)}$ do not involve the diagonal entries so it is an element of the elimination ideal.

In order to show that they are generated by minors of size $r+1$, it suffices to observe that for every ℓ, $\mathbf{E}_\ell^{(n,r)}$ is a subspace of the ideal generated by $\mathbf{E}_{\ell+1}^{(n,r)}$ because the image of $\wedge_{\rho\in\Omega_{\ell+1}^+} f_\rho$ lies in the ideal generated by the f_ρ's because the determinant of F_ℓ from (4.11) is a combination of the entries in the first row of F_ℓ with polynomial coefficients.

This shows that $\mathbf{E}_0^{(n,r)}$ is a subspace of the elimination ideal obtained from the ideal $I(\sigma_r)$ of minors of size $r+1$, by eliminating the diagonal entries. In particular, the elements of $\mathbf{E}_0^{(n,r)}$ are equations for $J(\sigma_r, L^S)$. $\square$

There is of course the possibility that the space $\mathbf{E}_0^{(n,r)}$ contains only the 0 polynomial. This happens for instance when $n=7, r=4$ (so that $\omega=3$); in this case $\mathbf{E}_1^{(7,4)}=0$ and so $\mathbf{E}_0^{(7,4)}=0$. The following two results suggest that this should not be *frequent*:

Proposition 4.11. *Fix n, r and let $\omega = 2(r+1)-n$. Then for every $\ell = \omega, \ldots, 0$, there exists a surjective map $\mathbf{E}_\ell^{(n+2,r+1)} \to \mathbf{E}_\ell^{(n,r)}$. In particular, if $\mathbf{E}_0^{(n,r)} \neq 0$, then $\mathbf{E}_0^{(n+2,r+1)} \neq 0$.*

Proof. For every ℓ, write ψ_ℓ for the maps of (4.10) at lever (n,r) and ψ'_ℓ for the ones at level $(n+2, r+1)$. We define a restriction map $ev^{(n+2,r+1)} : \mathbb{C}[x_j^i : i.j = 1, \ldots, n+2] \to \mathbb{C}[x_j^i : i,j = 1, \ldots, n]$ to be the evaluation at the matrix

$$\begin{pmatrix} x_1^1 & \cdots & x_n^1 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ x_1^n & \cdots & x_n^n & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Write $ev_\ell^{(n+2,r+1)} := ev^{(n+2,r+1)}|_{\mathbf{E}_\ell^{(n+2,r+1)}}$ for the restriction to $\mathbf{E}_\ell^{(n+2,r+1)}$. In this proof, denote $ev_\ell^{(n+2,r+1)}$ by ev_ℓ.

For every $\ell = \omega, \dots, 1$ consider the diagram

$$\begin{tikzcd} \bigwedge^{1+\binom{\omega}{\ell}} \mathbf{E}_\ell^{(n+2,r+1)} \arrow[r, "\psi'_\ell"] \arrow[d, "ev_\ell"'] & \mathbf{E}_{\ell-1}^{(n+2,r+1)} \arrow[d, "ev_{\ell-1}"] \\ \bigwedge^{1+\binom{\omega}{\ell}} \mathbf{E}_\ell^{(n,r)} \arrow[r, "\psi_\ell"] & \mathbf{E}_{\ell-1}^{(n,r)} \end{tikzcd} \tag{4.12}$$

We use reverse induction on ℓ to show that the vertical arrows are well-defined and surjective. If $\ell = \omega$, consider $I', J' \subseteq \{\omega+1, \dots, n\}$ with $I' \cap J' = \emptyset$ and $|I'| = |J'| = r + 1 - \omega$. Then

$$M_{1,\dots,\omega,J'}^{1,\dots,\omega,I'} = ev_\omega^{n+2,n}(M_{1,\dots,\omega,J',n+2}^{1,\dots,\omega,I',n+1}).$$

This shows that $ev_\omega^{n+2,n}$ surjects onto $\mathbf{E}_\omega^{(n,r)}$.

Now, by induction, we assume that the left vertical arrow of (4.12) is well defined and surjective. The diagram clearly commutes because the differentials in the definition of ψ_ℓ and ψ'_ℓ do not involve variables in the last two rows or the last two columns. The horizontal arrows are surjective by the definition of $\mathbf{E}_\ell^{(n,r)}$, therefore also the right vertical arrow has to be surjective. $\square$

Proposition 4.12. *The map $\psi_\omega : \bigwedge^2 \mathbf{E}_\omega^{(n,r)} \to \mathbf{E}_{\omega-1}^{(n,r)}$ is injective (so it is an isomorphism).*

Proof. Denote $\underline{\omega} = (1, \dots, \omega)$ and consider the basis of $\mathbf{E}_\omega^{(n,r)}$ given by minors of size $r+1$ of the form $M_{\underline{\omega}J}^{\omega I}$ with $I \cap J = \emptyset$ and $I \cup J = \{\omega+1, \dots, n\}$. Choose an ordering of the basis (namely an ordering of the pairs (I, J)) so that

$$M^{\underline{\omega} I_1}_{\underline{\omega} J_1} \wedge M^{\underline{\omega} I_2}_{\underline{\omega} J_2} \quad \text{for } (I_1, J_1) < (I_2, J_2)$$

is a basis of $\bigwedge^2 \mathbf{E}^{(n,r)}_{\omega}$. We obtain a spanning set for $\mathbf{E}^{(n,r)}_{\omega-1}$ by applying ψ_ω:

$$\mathbf{E}^{(n,r)}_{\omega-1} = \left\langle f^{I_1,I_2}_{J_1,J_2} : (I_1, J_1) < (I_2, J_2) \right\rangle$$

where

$$f^{I_1,I_2}_{J_1,J_2} := \det \begin{pmatrix} M^{\underline{\omega} I_1}_{\underline{\omega} J_1} & M^{\underline{\omega} I_2}_{\underline{\omega} J_2} \\ M^{I_1}_{J_1} & M^{I_2}_{J_2} \end{pmatrix}.$$

We prove that the $f^{I_1,I_2}_{J_1,J_2}$'s are linearly independent. Consider coefficients $a^{I_1,I_2}_{J_1,J_2}$ such that

$$\sum_{(I_1,J_1)<(I_2,J_2)} a^{I_1,I_2}_{J_1,J_2} \cdot f^{I_1,I_2}_{J_1,J_2} = 0; \tag{4.13}$$

we will prove $a^{I_1,I_2}_{J_1,J_2} = 0$ for every $(I_1, J_1) < (I_2, J_2)$.

Consider the result of the differentiation of (4.13) by the monomial $x^1_1 \cdots x^{\omega-1}_{\omega-1}$; denote by $g^{I_1,I_2}_{J_1,J_2}$ the derivative of $f^{I_1,I_2}_{J_1,J_2}$, so that

$$g^{I_1,I_2}_{J_1,J_2} := \det \begin{pmatrix} M^{\omega I_1}_{\omega J_1} & M^{\omega I_2}_{\omega J_2} \\ M^{I_1}_{J_1} & M^{I_2}_{J_2} \end{pmatrix}$$

and

$$\sum_{(I_1,J_1)<(I_2,J_2)} a^{I_1,I_2}_{J_1,J_2} \cdot g^{I_1,I_2}_{J_1,J_2} = 0. \tag{4.14}$$

First notice that $g^{I_1,I_2}_{J_1,J_2} \neq 0$ for every $(I_1, J_1) < (I_2, J_2)$: indeed, if that was the case, then $M^{\omega I_1}_{\omega J_1} M^{I_2}_{J_2} = M^{\omega I_2}_{\omega J_2} M^{I_1}_{J_1}$, but by the unique factorization in $\mathbb{C}[x^i_j : i, j = 1, \dots, n]$ this would imply $(I_1, J_1) = (I_2, J_2)$ because minors are irreducible polynomials.

Fix (I_1, J_1) and (I_2, J_2). We will show $a^{I_1,I_2}_{J_1,J_2} = 0$. Let $I^* := I_1 \cap I_2$ and $J^* := J_1 \cap J_2$. Notice that $|I^*| = |J^*|$ and $I^* \cap J^* = \emptyset$. Write $I^* = \{i^*_1, \dots, i^*_p\}$ and $J^* = \{j^*_1, \dots, j^*_p\}$. Notice that $I_1 \smallsetminus I^* = J_2 \smallsetminus J^*$ (call this set K) and $I_2 \smallsetminus I^* = J_1 \smallsetminus J^*$ (call this set L), with $|K| = |L|$, $K \cap L = \emptyset$ and suppose $K = \{k_1, \dots, k_q\}$ and $L = \{\ell_1, \dots, \ell_q\}$.

Differentiate (4.14) by the monomial

$$(x^{i^*_1}_{j^*_1} \cdots x^{i^*_p}_{j^*_p})^2 \cdot (x^{\omega}_{k_1} x^{\ell_1}_{k_2} \cdots x^{\ell_q}_{\omega}) \cdot (x^{k_1}_{\ell_1} \cdots x^{k_q}_{\ell_q}).$$

The result of the differentiation is, up to scale, $a^{I_1,I_2}_{J_1,J_2}$. This shows that $a^{I_1,I_2}_{J_1,J_2} = 0$.

We conclude that the $f^{I_1,I_2}_{J_1,J_2}$'s form a basis for $\mathbf{E}^{(n,r)}_{\omega-1}$, so the map ψ_ω is injective. $\square$

4.3 Representation theory of the determinantal sequence

In this section, we study the maps defined in (4.10) and the spaces $\mathbf{E}^{(n,r)}_\ell$ via the representation theory of the symmetric group.

The symmetric group $\mathfrak{S}_n$ acts on Mat_n via simultaneous permutation of rows and columns (namely via conjugation by a permutation matrix). The varieties $\sigma^{(n)}_r$ as well as the subspace of diagonal matrices are $\mathfrak{S}_n$-varieties. Therefore, the join $J(\sigma^{(n)}_r, L^S)$ is a $\mathfrak{S}_n$-variety as well.

The action of $\mathfrak{S}_n$ on Mat_n induces naturally an action on Mat^*_n and therefore on the spaces of polynomials $S^q\mathrm{Mat}^*_n$. Since $J(\sigma_r, L^S)$ is a $\mathfrak{S}_n$-variety, the subspace of $S^q\mathrm{Mat}^*_n$ that vanish on $J(\sigma_r, L^S)$ is a $\mathfrak{S}_n$-subrepresentation of $S^q\mathrm{Mat}^*_n$.

The space $\mathbf{E}_\omega^{(n,r)}$ of (4.9) is preserved by the action of the subgroup $\mathfrak{S}_\omega \times \mathfrak{S}_{n-\omega} \subseteq \mathfrak{S}_n$.

The following Lemma, whose proof is immediate, shows that the maps ψ_ℓ commute with the action of $\mathfrak{S}_\omega \times \mathfrak{S}_{n-\omega}$. This implies that every $\mathbf{E}_\ell^{(n,r)}$ is a representation for $\mathfrak{S}_\omega \times \mathfrak{S}_{n-\omega}$ as the intuition suggests.

Lemma 4.13. *For $\ell = \omega, \dots, 1$, the map $\psi_\ell : \mathbf{E}_\ell^{(n.r)} \to \mathbf{E}_{\ell-1}^{(n.r)}$ is $\mathfrak{S}_\omega \times \mathfrak{S}_{n-\omega}$-equivariant.*

Now consider the map $ev_\ell^{n+2,r+1}$ defined in the proof of Proposition 4.11. Notice that $\mathfrak{S}_\omega \times \mathfrak{S}_{n-\omega}$ acts on $\mathbf{E}_\ell^{(n+2,r+1)}$ via the restriction of the action of $\mathfrak{S}_\omega \times \mathfrak{S}_{n+2-\omega}$. We have the following easy, but important, result

Lemma 4.14. *For $\ell = \omega, \dots, 1$, the map $ev_\ell^{n+2,r+1} : \mathbf{E}_\ell^{(n+2,r-1)} \to \mathbf{E}_\ell^{(n,r)}$ is $\mathfrak{S}_\omega \times \mathfrak{S}_{n-\omega}$-equivariant.*

In particular, Lemma 4.13 and Lemma 4.14 show that the diagram in (4.12) commutes with the action of $\mathfrak{S}_\omega \times \mathfrak{S}_{n-\omega}$. It is immediate to verify that the action of $\mathfrak{S}_\omega$ is trivial, since it is trivial on $\mathbf{E}_\omega^{(n,r)}$ and all the maps in (4.12) are surjective.

The action of $\mathfrak{S}_{n-\omega}$ is more complicated. Notice that $n-\omega = 2(r+1-\omega)$ is even; write $m := r+1-\omega$ and denote $\mathfrak{S}_{n-\omega}$ as $\mathfrak{S}_{2m}$. The following result describes how $\mathfrak{S}_{2m}$ acts on $\mathbf{E}_\omega^{(n,r)}$.

Proposition 4.15. *As a $\mathfrak{S}_{2m}$-representation*

$$\mathbf{E}_\omega^{(n,r)} \simeq \mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_m}^{\mathfrak{S}_{2m}}([1^m] \otimes [1^m]) = [1^{2m}] \oplus [2, 1^{2m-2}] \oplus \cdots \oplus [2^m].$$

Proof. Consider $f := M_{1,\dots,\omega,\omega+m+1,\dots,n}^{1,\dots,\omega,\omega+1,\dots,\omega+m}$. It is an element of $\mathbf{E}_\omega^{(n,r)}$. A pair of permutations $(\sigma, \tau) \in \mathfrak{S}_m \times \mathfrak{S}_m \subseteq \mathfrak{S}_{2m}$ acts on the line $\langle f \rangle$ by multiplication by the sign of $(-1)^\sigma(-1)^\tau$. Therefore $\langle f \rangle = [1^m] \otimes [1^m]$. Representatives for the left cosets of

$\mathfrak{S}_m \times \mathfrak{S}_m$ in $\mathfrak{S}_{2m}$ permute the minors $M^{1,\ldots,\omega,I}_{1,\ldots,\omega,J}$.

The second inequality follows by Pieri's rule (Lemma 2.32). $\square$

From Proposition 4.12, we have that ψ_ω is an isomorphism of $\mathfrak{S}_{2m}$-modules between $\bigwedge^2 \mathbf{E}^{(n,r)}_\omega$ and $\mathbf{E}^{(n,r)}_{\omega-1}$. In particular, we obtain

$$\mathbf{E}^{(n,r)}_{\omega-1} = \bigwedge^2 \Big([1^{2m}] \oplus [2, 1^{2m-2}] \oplus \cdots \oplus [2^m] \Big).$$

In the next paragraph we focus on a particular case: we prove that when $\omega = 2(r+1) - n = 2$, the space of equations $\mathbf{E}^{(n,r)}_0$ contains a $\mathfrak{S}_{n-2}$ invariant.

Invariant equations for $\omega = 2$

In this paragraph, we restrict to the case $\omega = 2$. Let n, r such that $\omega = 2(r+1) - n = 2$, so that $n - \omega = n - 2 = 2(r-1)$, giving $m = r - 1$; in particular n is even. The sequence defined in (4.10) consists of two maps, that are $\mathfrak{S}_{2m}$-equivariant

$$\bigwedge^3 \left(\bigwedge^2 \mathbf{E}^{(n,r)}_2 \right) \xrightarrow{\psi_2^{\wedge 3}} \bigwedge^3 \mathbf{E}^{(n,r)}_1 \xrightarrow{\psi_1} \mathbf{E}^{(n,r)}_0. \tag{4.15}$$

We will prove in this section that $\mathbf{E}^{(n,r)}_0$ contains an invariant under the action of $\mathfrak{S}_{2m}$. We will realize the invariant as follows. From Proposition 4.15, we have

$$\mathbf{E}^{(n,r)}_2 = \operatorname{Ind}^{\mathfrak{S}_{2m}}_{\mathfrak{S}_m \times \mathfrak{S}_m} ([1^m] \otimes [1^m]) = [1^{2m}] \oplus [2, 1^{2m-2}] \oplus \cdots \oplus [2^m].$$

From standard manipulations with exterior powers (see e.g. [Lan12], Formula (6.7.2) at p.157), we obtain

$$\bigwedge^2 \mathbf{E}_2^{(n,r)} \supseteq \bigwedge^2[2^m] \oplus \bigwedge^2[2, 1^{2m-2}].$$

We will prove $\bigwedge^2[2^m] \supseteq [2m-3, 1, 1, 1]$ and $\bigwedge^2[2, 1^{2m-2}] = [2m-2, 1, 1]$.

In particular, using (6.7.2) of [Lan12] again, we obtain

$$\begin{aligned}\bigwedge^3 \bigwedge^2 \mathbf{E}_2^{(n,r)} &\supseteq \bigwedge^3\big([2m-3,1,1,1] \oplus [2m-2,1,1]\big) \supseteq \\ &\supseteq \bigwedge^2[2m-3,1,1,1] \otimes [2m-2,1,1]\end{aligned}$$

and the latter contains a $\mathfrak{S}_{2m}$-invariant. We will show that this invariant does not map to 0 via the composition $\psi_1 \circ (\psi_2^{\wedge 3})$.

Lemma 4.16. *For every $m \geq 2$, $[2m-3,1,1,1]$ occurs in the decomposition of $\bigwedge^2[2^m]$ as $\mathfrak{S}_{2m}$-representation.*

Proof. Consider the standard Young tableaux of shape (2^m) defined by

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \vdots & \vdots \\ \hline 2m-1 & 2m \\ \hline\end{array}\,, \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \vdots & \vdots \\ \hline 2m-1 & 2m \\ \hline\end{array}\,.$$

From 2.30, we have $\mathcal{Y}_{T_1}[2^m]$ and $\mathcal{Y}_{T_2}[2^m]$ are distinct one-dimensional subspaces of $[2^m]$; let v_i be an element of $\mathcal{Y}_{T_i}[2^m]$.

Consider the standard Young tableau of shape $(2m-3,1,1,1)$ defined by

$$P^m = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 5 & \cdots & 2m-2 & 2m-1 & 2m \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}.$$

Let $\mathcal{Y}_{P^m}$ be the corresponding Young symmetrizer. We will show that $\mathcal{Y}_{P^m}(v_1 \wedge v_2) \neq 0$.

We use induction on m. For $m = 2$, the claim is true and can be checked by an explicit calculation (see Appendix C).

For $m \geq 3$ consider the restriction of the action of $\mathfrak{S}_{2m}$ to $\mathfrak{S}_{2m-2}$ on $[2^m]$. By Pieri's rule, we have

$$\mathrm{Res}^{\mathfrak{S}_{2m}}_{\mathfrak{S}_{2m-2}}([2]) = [2^{m-1}] \oplus [2^{m-2},1,1].$$

Define $\pi : [2^m] \to [2^{m-1}]$ to be the projection on the first summand. Let $\widehat{T}$ be Young tableaux obtained from a tableau T by removing the last row. We have

$$\pi(v_T) = \begin{cases} v_{\widehat{T}} & \text{if the last row of } T \text{ is } (2m-1,2m), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand

$$\bigwedge^2 [2^m] = \bigwedge^2 [2^{m-1}] \oplus [2^{m-1}] \otimes [2^{m-2},1,1] \oplus \bigwedge^2 [2^{m-2},1,1].$$

Denote by $\pi^\wedge : \bigwedge^2 [2^m] \to \bigwedge^2 [2^{m-1}]$ the projection. We show $\pi^\wedge \circ \mathcal{Y}_{P^m}(v_1 \wedge v_2) = \mathcal{Y}_{P_{m-1}} \circ \pi^\wedge(v_1 \wedge v_2)$.

It is clear that $\mathcal{Y}_{P^{m-1}} \circ \pi^\wedge(v_1 \wedge v_2) = \pi^\wedge \circ \mathcal{Y}_{P^{m-1}}(v_1 \wedge v_2)$ because $\mathcal{Y}_{P^{m-1}}$ does not affect the last row of the diagrams associated to v_1 and v_2.

Notice that P^m and P^{m-1} have the same symmetrizer of the columns, that is $b_{P^m} = b_{P^{m-1}} = \sum_{\tau \in \mathfrak{S}_4} (-1)^\tau \delta_\tau$. Let a_{P^m} and $a_{P^{m-1}}$ be the symmetrizers of the rows of P^m and P^{m-1} respectively. We have $a_{P^m} = \sum_{\sigma \in \mathfrak{S}_{\{1,5,\ldots,2m\}}} \sigma$ and similarly $a_{P^{m-1}}$. Here, for a finite set Σ, we denote by $\mathfrak{S}_\Sigma$ the symmetric group acting on the set Σ. Using $\mathrm{id}, (2m-1, 2m)$ and $(i, 2m-1)(j, 2m)$ (for $i, j \in \{1, 5, \ldots, 2m\}$, not both in $\{2m-1, 2m\}$ as a set of representatives for the left cosets of $\mathfrak{S}_{\{1,5,\ldots,2m-2\}}$ in $\mathfrak{S}_{\{1,5,\ldots,2m\}}$, we have

$$a_{P^m} = a_{P^{m-1}} + \delta_{(2m-1,2m)} a_{P^{m-1}} + \left(\sum_{\substack{i,j=1,5,\ldots,2m \\ i \neq j, \{i,j\} \neq \{2m-1,2m\}}} \delta_{(i,2m-1)(j,2m)} \right) a_{P^{m-1}};$$

so we have

$$\begin{aligned} \mathcal{Y}_{P^m} &= b_{P^m} a_{P^m} = b_{P^{m-1}} a_{P^{m-1}} + \delta_{(2m-1,2m)} b^{P^{m-1}} a_{P^{m-1}} + G = \\ &= (\delta_{\mathrm{id}} + \delta_{(2m-1,2m)})) \mathcal{Y}_{P^{m-1}} + G \end{aligned}$$

where $G = \left(\sum_{\substack{i,j=1,5,\ldots,2m \\ i \neq j, \{i,j\} \neq \{2m-1,2m\}}} \delta_{(i,2m-1)(j,2m)} \right) \mathcal{Y}_{P^{m-1}}$.

We can observe that $\pi(G(v_1 \wedge v_2)) = 0$. On the other hand $\mathcal{Y}_{P^{m-1}} \circ \pi^\wedge(v_1 \wedge v_2) = \pi^\wedge \circ \mathcal{Y}_{P^{m-1}}(v_1 \wedge v_2)$ as observed above. Moreover, an elementary application of the straightening algorithm provides $\delta_{(2m-1,2m)} v_1 = v_1 + w$ where $\pi(w) = 0$ and similarly for v_2; therefore $\pi^\wedge(\delta_{(2m-1,2m)} \mathcal{Y}_{P^{m-1}}(v_1 \wedge v_2)) = \pi^\wedge(\mathcal{Y}_{P^{m-1}}(v_1 \wedge v_2)) = \mathcal{Y}_{P^{m-1}} \circ \pi^\wedge(v_1 \wedge v_2)$.

This shows that the image of $v_1 \wedge v_2$ via $\mathcal{Y}_{P^{m-1}} \circ \pi^\wedge$ is the same as its image via

$\pi^{\wedge} \circ \mathcal{Y}_{P^m}$.

Moreover $\pi^{\wedge}(v_1 \wedge v_2) = v_1' \wedge v_2'$ where v_1', v_2' are the vectors in $[2^{m-2}]$ associated to the Young tableaux obtained from T_1, T_2 removing the last row. So, by inductive hypothesis $\mathcal{Y}_{P^{m-1}} \circ \pi^{\wedge}(v_1 \wedge v_2) \neq 0$. This shows that $\mathcal{Y}_{P^m}(v_1 \wedge v_2) \neq 0$ because its image via $\pi^{\wedge}$ is nonzero.

We conclude that $[2m-3, 1, 1, 1]$ appears in the decomposition of $\bigwedge^2[2^m]$ as $\mathfrak{S}_{2m}$-representation. $\square$

Lemma 4.16 and Proposition 4.12 imply that $\mathbf{E}_1^{(n,r)}$ contains a copy of $[2m-3, 1, 1, 1]$. Moreover, by Schur's Lemma, we deduce that the map π in the proof of Lemma 4.16 coincides, up to scale, with (the restriction to $[2^{m-1}]$ of) the map $ev_2^{(n,r)}$ of Proposition 4.9, because they both define an equivariant map between the same two irreducible $\mathfrak{S}_{2m}$-representations.

Lemma 4.17. *For every $m \geq 1$, we have $\bigwedge^2[2, 1^{2m-2}] = [2m-2, 1, 1]$.*

Proof. We use the following two facts (see e.g. [FH91], Sec. 4.1, 4.2): for any p, q with $q \leq p-1$, $\bigwedge^q[p-1, 1] = [p-q, 1^q]$; for every partition $\pi \vdash d$, $\pi \otimes [1^d] = \pi'$, where π' is the conjugate partition of π (that is the partition whose Young diagram is the transpose about the main diagonal of the diagram of π).

In particular, we have

$$[2, 1^{2m-2}] = [2m-1, 1] \otimes [1^{2m}],$$

so that, using Formula (6.7.4), p.157 of [Lan12] (notice that $\bigwedge^2[1^{2m}] = 0$),

$$\begin{aligned}\bigwedge^2[2, 1^{2m-2}] &= \bigwedge^2([2m-1,1] \otimes [1^{2m}]) = \\ &= \bigwedge^2[2m-1,1] \otimes S^2[1^d] = \\ &= [2m-2,1,1] \otimes [d] = [2m-2,1,1].\end{aligned}$$

$\square$

Lemma 4.18. *For every $m \geq 3$, the module $\bigwedge^2[2m-3,1,1,1]$ contains a copy of $[2m-2,1,1]$.*

Proof. We briefly outline the proof, that is essentially the same as Lemma 4.16.

Consider standard Young tableaux T_1 and T_2 of shape $(2m-3,1,1,1)$ as follows:

$$T_1 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 5 & 6 & \cdots & 2m-1 & 2m \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

$$T_2 = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 4 & 6 & \cdots & 2m-1 & 2m \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array},$$

so that $v_i \in \mathcal{Y}_{T_i}[2m-3,1,1,1]$ (for $i = 1, 2$ are two linearly independent vectors in $[2m-3,1,1,1]$.

Let P^m be the standard Young tableaux of shape $(2m-2,1,1)$

$$P^m = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & \cdots & 2m-1 & 2m \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array},$$

and let $\mathcal{Y}_{P^m}$ be its Young symmetrizer. Using an inductive argument as in Lemma 4.16, we can show that $\mathcal{Y}_{P^m}(v_1 \wedge v_2) \neq 0$. The base of the induction is an explicit calculation (see Appendix C).

The inductive step can be performed with the exact same calculation as Lemma 4.16. $\square$

It is an elementary fact, that follows from Schur's Lemma, that for every partition $\lambda \vdash d$, $[\lambda] \otimes [\lambda]$ contains exactly one copy of $[d]$. Indeed, $[\lambda] \otimes [\lambda] = \mathrm{Hom}([\lambda]^*, [\lambda]) = \mathrm{Hom}([\lambda], [\lambda])$, because $[\lambda] \simeq [\lambda]^*$. The $\mathfrak{S}_d$-invariant subspace is the subspace $\mathrm{Hom}_{\mathfrak{S}_d}([\lambda], [\lambda])$ of $\mathfrak{S}_d$-equivariant homomorphism, that is one-dimensional by Schur's Lemma.

Finally, we are able to prove

Theorem 4.19. *For every $n = 2(m+1) \geq 4$ even and $r = m+1$, we have $\omega = 2$ and $\mathbf{E}_0^{(n,r)}$ contains a $\mathfrak{S}_{2m}$-invariant polynomial.*

Proof. The space $\mathbf{E}_0^{(n,r)}$ is contained, as $\mathfrak{S}_{2m}$-module, into $\bigwedge^3 \mathbf{E}_1^{(n,r,)}$. Lemma 4.16 and Lemma 4.17, with Proposition 4.12, show that $\mathbf{E}_1^{(n,r)}$ contains a copy of $[2m-3,1,1,1]$

and a copy of $[2m-2,1,1]$. Therefore

$$\begin{aligned}\bigwedge^3 \mathbf{E}_1^{(n,r,)} \supseteq \bigwedge^3\Big([2m-3,1,1,1] \oplus [2m-2,1,1]\Big) \supseteq \\ \supseteq [2m-2,1,1] \otimes \bigwedge^2[2m-2,1,1,1] \supseteq \\ [2m-2,1,1] \otimes [2m-2,1,1] \supseteq [2m]\end{aligned}$$

where we used again Formula (6.7.2) in [Lan12] and Lemma 4.18.

It suffices to show that this invariant is not mapped to 0 via $\psi_1^{(n,r)} : \mathbf{E}_1^{(n,r)} \to \mathbf{E}_0^{(n,r)}$. This can be shown via an explicit calculation for $m = 3$ (that corresponds to the case $n = 4$) (see Appendix C). The general statement follows via an induction argument and Proposition 4.11. $\square$

5. RIGID MATRICES

In this chapter we use the tests for non membership in $J(\sigma_r, L^S)$ (that is the equations that we developed in Chapter 4) to determine matrices that are rigid in the restricted range in which we are able to evaluate our equations. The results from Section 5.1 are from [GHIL16], while the ones of Section 5.2 are original.

5.1 Maximal Rigidity and Border Rigidity

Definition 5.1. We say that $A \in \mathrm{Mat}_n$ is *maximally r-rigid* if $\mathrm{Rig}_r(A) = (n-r)^2$. We say that $A \in \mathrm{Mat}_n$ is *maximally r-border rigid* if $A \notin \mathcal{R}[n, r, s]$ (or equivalently $\underline{\mathrm{Rig}}_r(A) = (n-r)^2$) whenever $\mathcal{R}[n, r, s]$ is a proper subvariety of Mat_n, so for $s < (n-r)^2$.

Notice that if A is maximally r-border rigid then it is maximally r-rigid. In order to prove that A is maximally r-border rigid, we only need equations for $\mathcal{R}[n, r, s]$ when it is a hypersurface, so when $s = (n-r)^2 - 1$. In particular if $f_1, \ldots, f_r$ are equations for the irreducible components of the hypersurface $\mathcal{R}[n, r, (n-r)^2 - 1]$, then a given A is maximally r-border rigid if and only if $f_i(A) \neq 0$ for every $i = 1, \ldots, r$.

Both [Val77] and [Lok09] suggest that one should study the rigidity of families of matrices commonly used in mathematics; we focus on Cauchy matrices and Vandermonde matrices.

Definition 5.2. Define the *Cauchy map*:

$$
\begin{aligned}
Cau_n : \mathbb{C}^n \times \mathbb{C}^n &\dashrightarrow \mathrm{Mat}_n \\
((w_i)_{i=1,\dots,n}, (z_j)_{j=1,\dots,n}) &\mapsto \left(\frac{1}{w_i + z_j} \right)_{ij=1,\dots,n}.
\end{aligned}
$$

The closure of the image of Cau_n in Mat_n is an algebraic variety called the variety of *Cauchy matrices*: denote it by $\mathcal{C}au_n$.

The Cauchy variety $\mathcal{C}au_n$ is invariant under permutations of rows and columns; this means that it is a $\mathfrak{S}_n \times \mathfrak{S}_n$-variety.

Proposition 5.3. *A general Cauchy matrix is both maximally* 1*-border rigid and maximally* $(n-2)$*-border rigid.*

Proof. We use the equations from Theorem 4.7 and Theorem 4.8.

First we prove that a general element $A \in \mathcal{C}au_n$ is not in $\mathcal{R}[n, 1, n^2 - 2n]$. Up to permutations of rows and columns, we consider the irreducible components of $\mathcal{R}[n, 1, n^2 - 2n]$ having equation

$$
x_1^1 \cdots x_k^k - x_{\sigma(1)}^1 \cdots x_{\sigma(k)}^k = 0,
$$

for some k-cycle σ of $\mathfrak{S}_n$.

Since the x_j^i's are not identically 0 on $\mathcal{C}au_n$, rewrite the equations as

$$
\frac{1}{x_1^1} \cdots \frac{1}{x_k^k} = \frac{1}{x_{\sigma(1)}^1} \cdots \frac{1}{x_{\sigma(k)}^k}.
$$

Evaluation at $\mathcal{C}au_n$ gives

$$(z_1 + w_1) \cdots (z_k + w_k) = (z_1 + w_{\sigma(1)}) \cdots (z_k + w_{\sigma(k)})$$

On the left-hand side, the term $z_1 \cdots z_{k-1} w_k$ appears; this term does not appear on the right-hand side because $w_k \neq w_{\sigma(k)}$. This proves that a general Cauchy matrix is maximally 1-border rigid.

To prove that a general element $A \in \mathcal{C}au_n$ is not in $\mathcal{R}[n, n-2, 3]$, first observe that every submatrix of a Cauchy matrix is a Cauchy matrix; moreover, we have (see e.g. [Sch59])

$$\det(Cau_n((w_i), (z_j))) = \frac{\prod_{i<j}(w_i - w_j)(z_i - z_j)}{\prod_{ij}(w_i + z_j)},$$

that is non-zero if the w_i's are distinct and the z_j's are distinct. In particular, all the minors of an invertible Cauchy matrix are non-zero, so A certainly does not belong to the components of $\mathcal{R}[n, n-2, 3]$ defined by avoiding minors.

Up to permutation of rows and columns, consider the irreducible component of $\mathcal{R}[n, n-2, 3]$ having equation

$$\det \begin{pmatrix} M^{13(123)^c}_{12(123)^c} & M^{12(123)^c}_{13(123)^c} \\ M^{3(123)^c}_{2(123)^c} & M^{2(123)^c}_{3(123)^c} \end{pmatrix} = 0$$

as in (4.5). Evaluation at a generic element of $\mathcal{C}au_n$ provides, after cancellation

$$\det \begin{pmatrix} \frac{(w_1 - w_3)(z_1 - z_2)}{(w_1 + z_2)(w_3 + z_1)} & \frac{(w_1 - w_2)(z_1 - z_3)}{(w_1 + z_3)(w_2 + z_1)} \\ 1 & 1 \end{pmatrix} = 0.$$

This does not hold for generic choice of $w_1, w_2, w_3, z_1, z_2, z_3$, therefore a generic element of $\mathcal{C}au_n$ does not satisfy the equation. This shows that a generic Cauchy matrix is maximally $(n-2)$-border rigid. $\square$

Definition 5.4. Define the *Vandermonde map*:

$$\begin{aligned} Van_n : \mathbb{C}^n &\to \mathrm{Mat}_n \\ (z_j)_{j=1,\dots,n} &\mapsto \left(z_j^{i-1}\right)_{ij=1,\dots,n}. \end{aligned}$$

The closure of the image of Van_n in Mat_n is an algebraic variety called the variety of *Vandermonde matrices*, or *Vandermonde variety*: denote it by $\mathcal{V}and_n$.

The Vandermonde variety $\mathcal{V}and_n$ is invariant under permutation of columns; this means it is a $\mathfrak{S}_n$-variety, where $\mathfrak{S}_n$ acts by right multiplication by permutation matrices.

Notice that the DFT matrix DFT_n is a particular Vandermonde matrix, the image via Van_n of $(\omega^j)_{j=0,\dots,n-1}$, where ω is a primitive n-th root of 1.

Proposition 5.5. *A general Vandermonde matrix is both maximally* 1*-border rigid and maximally* $(n-2)$*-border rigid.*

Proof. Let $A \in \mathcal{V}and_n$ be a general element. We prove $A \notin \mathcal{R}[n, 1, n^2 - 2n]$. Up to permutation of columns, consider the component having equation

$$x_1^{i_1} \cdots x_k^{i_k} - x_{\sigma(1)}^{i_1} \cdots x_{\sigma(k)}^{i_k} = 0$$

for some choice of rows $i_1, \dots, i_k$ and some k-cycle $\sigma \in \mathfrak{S}_n$. When we evaluate this equation at $A = Vand_n(z_1, \ldots, z_n)$, we obtain

$$z_1^{i_1-1} \cdots z_k^{i_k-1} - z_{\sigma(1)}^{i_1-1} \cdots z_{\sigma(k)}^{i_k-1} = 0$$

where now the upper indices are exponents. Since the i_j's are distinct and σ is not the identity, this expression cannot be identically 0 as a polynomial in $z_1, \ldots, z_n$. Therefore, we conclude that A does not satisfy the equations for $\mathcal{R}[n, 1, n^2 - 2n]$, so it is maximally 1-border rigid.

To prove that A is maximally $(n-2)$-border rigid, first observe that all the minors of a generic Vandermonde matrix are non-zero (they are so-called alternating polynomials, see e.g. [Mac98], Sec. I.3, p.40). In particular, A does not belong to the components of $\mathcal{R}[n, n-2, 3]$ defined by avoiding minors. Up to permutation of columns, consider the components having equation

$$\det \begin{pmatrix} M_{12(123)^c}^{ij(ijk)^c} & M_{13(123)^c}^{ik(ijk)^c} \\ M_{2(123)^c}^{j(ijk)^c} & M_{3(123)^c}^{k(ijk)^c} \end{pmatrix} = 0$$

for some i, j, k distinct row indices. Let B be the evaluation of the above 2×2 matrix at $A = Van_n(z_1, \ldots, z_n)$: then $\det(B)$ is a polynomial in $z_1, \ldots, z_n$. We show that it is not identically 0.

First suppose $2 \notin \{i, j, k\}$ so $2 \in (i, j, k)^c$. Regard $\det(B)$ as a polynomial in z_1, z_2 and consider the coefficient of the monomial $z_1 z_2$ of degree 2: it has coefficient 0 in the product $b_{11} b_{22}$ because b_{22} does not depend on z_1, z_2 and b_{11} has coefficient 0 in $z_1 z_2$; on the other hand, $z_1 z_2$ has a non-zero coefficient in the product $b_{12} b_{21}$, given by the product of two minors; therefore $\det(B) \neq 0$ if $2 \notin \{i, j, k\}$.

If $2 \in \{i, j, k\}$, without loss of generality suppose $i = 2$. Regard $\det(B)$ as a polynomial in z_2 and observe that the coefficient of the linear term is non-zero because it is the product of two minors.

This shows that the equation given by $\det(B)$ is not identically 0 on $\mathcal{V}and_n$, so a generic Vandermonde matrix is maximally $(n-2)$-border rigid. $\square$

The following two results suggest that matrices with strong symmetry cannot be maximally rigid.

Lemma 5.6. *If A is symmetric or symmetric about the anti-diagonal then A is not maximally 1-border rigid.*

Proof. Let S be the set of $n^2 - 2n$ entries omitting x^i_{i+1} (for $i = 1, \ldots, n-1$), x^n_1, x^1_n, x^{i+1}_i (for $i = 1, \ldots, n-1$).

Then $J(\sigma_1, L^S)$ is a hypersurface of equation

$$x^1_2 \cdots x^n_1 - x^2_1 \cdots x^1_n$$

that is satisfied by every symmetric matrix.

The argument is similar for a matrix that is symmetric about the anti-diagonal. $\square$

In particular, DFT_n is not maximally 1-border rigid. Moreover

Lemma 5.7. *If A is a generic matrix with an eigenvalue with multiplicity k, then $\mathrm{Rig}_{n-k}(A) \leq n$.*

Proof. Let c be an eigenvalue of A with multiplicity k and let $B = A - cI$ (where I

is the identity matrix). Since A is generic, $\mathrm{rank}(B) = n - k$ and $A = B + C$ with C diagonal (and so n-sparse). This shows $\mathrm{Rig}_{n-k}(A) \leq n$. □

5.2 Barak's problem for $\varepsilon \leq 1/2$

In this section we give a simple answer to Problem 1.4 when $\varepsilon < 1/2$. A non-explicit solution is also given when $\varepsilon = 1/2$.

Proposition 5.8. *Let $r \leq n/2 - 1$. Let A be a matrix such that all minors of size $(r + 1)$ are non-zero. Then A is not in $\mathcal{D}[n, r]$.*

Proof. Since $r \leq n/2 - 1$, for every σ there is at least a minor of size $r + 1$ that avoids the entries of S^σ, so that it is an equation for $J(\sigma_r, L^{S^\sigma})$. If all $(r + 1)$ are non-zero, then for every $\sigma \in \mathfrak{S}_n$ at least one equations of $J(\sigma_r^{(n)}, L^{S^\sigma})$ is non-zero, namely $A \notin \mathcal{D}[n, r]$. □

Proposition 5.8 provides a very simple method to obtain a solution to Problem 1.4 when $\varepsilon < 1/2$. In particular a sequence of general Cauchy matrices or a sequence of general Vandermonde matrices are non-explicit solutions to Problem 1.4. Moreover, since every minor of an invertible Cauchy matrix is nonzero, we can easily construct explicit sequences of Cauchy matrices that solve Problem 1.4. For instance, let w_i, z_j be both the sequence of natural numbers. Define for every n $A_n := Cau_n((w_i)_i, (z_j)_j)$. Then $\{A_n\}$ is an explicit solution to Problem 1.4 when $\varepsilon < 1/2$. Similarly, sequences of totally positive matrices (in the sense of [FZ00]) provide explicit solutions in this range.

We provide a non-explicit solution for the case $\varepsilon = 1/2$. If $n = 2r$ then we can apply the method that we developed in Chapter 4. In this range, we have $\omega = 2(r+1)-n =$

2. We will prove that a generic Cauchy matrix is not in the variety $\mathcal{D}[n,r]$ in this range.

Theorem 5.9. *If $r \geq 3$, a generic Cauchy matrix does not belong to $\mathcal{D}[n,r]$ if $n = 2r$.*

Proof. Because of the $\mathfrak{S}_n \times \mathfrak{S}_n$ invariance of $\mathcal{C}au_n$, without loss of generality we can consider the case where S is diagonal.

We will consider the equation of $J(\sigma_r, L^S)$ that arises as follows. Fix three distinct pairs $(I_1,J_1),(I_2,J_2),(I_3,J_3)$ such that, for $\ell = 1,2,3$, $I_\ell, J_\ell \subseteq \{3,\dots,n\}$, $I_\ell \cap J_\ell = \emptyset$ and $|I_\ell| = |J_\ell| = r-1$. In particular, for $\ell = 1,2,3$, $M^{12I_\ell}_{12J_\ell}$ and $M^{12J_\ell}_{12I_\ell}$ are elements of $\mathbf{E}^{(n,r)}_2$.

Define $g_\ell := \psi^{(n,r)}_2\left(M^{12I_\ell}_{12J_\ell} \wedge M^{12J_\ell}_{12I_\ell}\right) \in \mathbf{E}^{(n,r)}_1$, and $f = \psi^{(n,r)}_1(g_1 \wedge g_2 \wedge g_3) \in \mathbf{E}^{(n,r)}_0$. We will show that for a suitable choice of I_1, I_2, I_3 (and a consequent choice of J_1, J_2, J_3), the equation f does not vanish on a generic Cauchy matrix.

Fix $I', J' \subseteq \{7,\dots,n\}$, disjoint of cardinality $r-3$, so that $I' \cup J' = \{7,\dots,n\}$; let

$$I_1 = \{3,4\} \cup I', \quad J_1 = \{5.6\} \cup J',$$
$$I_2 = \{3,5\} \cup I', \quad J_1 = \{4.6\} \cup J',$$
$$I_3 = \{3,6\} \cup I', \quad J_1 = \{4,5\} \cup J'.$$

Recall that every square submatrix of a Cauchy matrix is a Cauchy matrix itself and that

$$\det(Cau_n((w_i),(z_j))) = \frac{\prod_{i<j}(w_i - w_j)(z_i - z_j)}{\prod_{ij}(w_i + z_j)}.$$

Fix ℓ and write $I = I_\ell$, $J = J_\ell$ so that $I = \{p_1, p_2\} \cup I'$ and $J = \{q_1, q_2\} \cup J'$ with $\{p_1, p_2, q_1, q_2\} = \{3, 4, 5, 6\}$. Let

$$g := \psi_2^{(n,r)} \left(M_{12J}^{12I} \wedge M_{12I}^{12J} \right) = \det \begin{bmatrix} M_{12J}^{12I} & M_{12I}^{12J} \\ M_J^I & M_I^J \end{bmatrix}$$

so that, letting $\{k, k'\} = \{1, 2\}$, we have

$$\partial_k g = \frac{\partial}{\partial x_k^k} g = \det \begin{bmatrix} M_{k'J}^{k'I} & M_{k'I}^{k'J} \\ M_J^I & M_I^J \end{bmatrix}.$$

Let

$$Q_J^I = \frac{\prod_{(i,j) \in I \times J} (w_i + z_j)}{\prod_{i,i' \in I, i < i'} (w_{i'} - w_i) \prod_{j,j' \in J, j < j'} (z_{j'} - z_j)}.$$

and similarly Q_I^J. Also, for sets of indices K, L define

$$U_L^K = \frac{\prod_{k \in K} (w_k - w_1)(w_k - w_2) \prod_{\ell \in L} (z_\ell - z_1)(z_\ell - z_2)}{\prod_{k \in K} (w_k + z_1)(w_k + z_2) \prod_{\ell \in L} (w_1 + z_\ell)(w_2 + z_\ell)},$$
$$V_L^K(1) = \frac{\prod_{k \in K} (w_k - w_1) \prod_{\ell \in L} (z_\ell - z_1)}{\prod_{k \in K} (w_k + z_1) \prod_{\ell \in L} (w_1 + z_\ell)},$$

and similarly $V_L^K(2)$.

By restricting the minors to $\mathcal{C}au_n$, we have

$$M_{12J}^{12I} = Q_J^I \cdot \frac{(w_2 - w_1)(z_2 - z_1)}{(z_1 + w_1)(z_1 + w_2)(z_2 + w_1)(z_2 + w_2)} \cdot U_{q_1, q_2}^{p_1, p_2} \cdot U_{J'}^{I'}.$$
$$M_{kJ}^{kI} = Q_J^I \cdot \frac{1}{(z_k + w_k)} \cdot V_{q_1 q_2}^{p_1 p_2}(k) \cdot V_{J'}^{I'}(k)$$
$$M_J^I = Q_J^I,$$

and similar relations exchanging the roles of I and J.

From these relations, we obtain

$$
\begin{aligned}
g(A) = \psi_2^{(n,r)}(M_{12J}^{12I} \wedge M_{12I}^{12J}) = \\
= Q_J^I Q_I^J \cdot \frac{(w_2-w_1)(z_2-z_1)}{(z_1+w_1)(z_1+w_2)(z_2+w_1)(z_2+w_2)} \cdot \\
\cdot \det \begin{bmatrix} U_{q_1,q_2}^{p_1,p_2} \cdot U_{J'}^{I'} & U_{p_1,p_2}^{q_1,q_2} \cdot U_{I'}^{J'} \\ 1 & 1 \end{bmatrix}.
\end{aligned}
$$

Similarly, for $k = 1, 2$,

$$
(\partial_k g)(A) = Q_J^I Q_I^J \cdot \frac{1}{(z_{k'} + w_{k'})} \cdot \det \begin{bmatrix} V_{q_1 q_2}^{p_1 p_2}(k') \cdot V_{J'}^{I'}(k') & V_{p_1 p_2}^{q_1 q_2}(k') \cdot V_{I'}^{J'}(k') \\ 1 & 1 \end{bmatrix}.
$$

We specialize to a Cauchy matrix A with the property that $U_{J'}^{I'} = U_{I'}^{J'}$ and $V_{J'}^{I'}(k) = V_{I'}^{J'}(k)$ for $k = 1, 2$. To do this, fix a bijection $\tau : I' \to J'$ and let $w_1 = z_1, w_2 = z_2$ and $w_i = z_{\tau(i)}$ for $i \in I'$ and $w_j = z_{\tau^{-1}(j)}$ for $j \in J'$. Let w_ℓ and z_ℓ be generic for $\ell \in \{3, 4, 5, 6\}$. We obtain

$$
g(A) = Q_J^I Q_I^J \cdot C_{12} \cdot U_{J'}^{I'} \cdot \left(U_{q_1,q_2}^{p_1,p_2} - U_{p_1,p_2}^{q_1,q_2} \right),
$$

where $C_{12} := \dfrac{(w_2-w_1)(z_2-z_1)}{(z_1+w_1)(z_1+w_2)(z_2+w_1)(z_2+w_2)}$ and similarly

$$
(\partial_k g)(A) = Q_J^I Q_I^J \cdot \frac{1}{(z_{k'} + w_{k'})} \cdot V_{J'}^{I'}(k') \cdot \left(V_{q_1,q_2}^{p_1,p_2}(k') - V_{p_1,p_2}^{q_1,q_2}(k') \right).
$$

By applying these relations to the pairs (I_ℓ, J_ℓ) for $\ell = 1, 2, 3$, we obtain the equation

f as a determinant of a 3×3 matrix whose columns are, for $\ell = 1, 2, 3$

$$\begin{bmatrix} g_\ell \\ \partial_1 g_\ell \\ \partial_2 g_\ell \end{bmatrix} = \begin{bmatrix} Q^{I_\ell}_{J_\ell} Q^{J_\ell}_{I_\ell} \cdot C_{12} \cdot U^{I'}_{J'} \cdot \left(U^{p_1,p_2}_{q_1,q_2} - U^{q_1,q_2}_{p_1,p_2}\right) \\ Q^{I_\ell}_{J_\ell} Q^{J_\ell}_{I_\ell} \cdot \frac{1}{(z_2+w_2)} \cdot V^{I'}_{J'}(2) \cdot \left(V^{p_1,p_2}_{q_1,q_2}(2) - V^{q_1,q_2}_{p_1,p_2}(2)\right) \\ Q^{I_\ell}_{J_\ell} Q^{J_\ell}_{I_\ell} \cdot \frac{1}{(z_1+w_1)} \cdot V^{I'}_{J'}(1) \cdot \left(V^{p_1,p_2}_{q_1,q_2}(1) - V^{q_1,q_2}_{p_1,p_2}(1)\right) \end{bmatrix};$$

in particular, by multilinearity, we obtain

$$\begin{aligned} f(A) = &\left(\prod_{\ell=1,2,3} Q^{I_\ell}_{J_\ell} Q^{J_\ell}_{I_\ell}\right) \cdot C_{12} \cdot \frac{1}{(z_1+w_1)(z_2+w_2)} \cdot U^{I'}_{J'} \cdot V^{I'}_{J'}(1) \cdot V^{I'}_{J'}(2) \cdot \\ &\cdot \det \begin{bmatrix} U^{34}_{56} - U^{56}_{34} & U^{35}_{46} - U^{46}_{35} & U^{36}_{45} - U^{45}_{36} \\ V^{34}_{56}(2) - V^{56}_{34}(2) & V^{35}_{46}(2) - V^{46}_{35}(2) & V^{36}_{45}(2) - V^{45}_{36}(2) \\ V^{34}_{56}(1) - V^{56}_{34}(1) & V^{35}_{46}(1) - V^{46}_{35}(1) & V^{36}_{45}(1) - V^{45}_{36}(1) \end{bmatrix}. \end{aligned} \tag{5.1}$$

Notice that the factors on the first line of (5.1) are nonzero if the w_i's are distinct among themselves and the z_j's are distinct among themselves. Moreover, notice that the matrix at the second line of (5.1) does not depend on w_i or z_j for $i, j \geq 7$.

In particular, it suffices to check that this determinant is non zero for a 6×6 Cauchy matrix $Cau_6(\mathbf{w}, \mathbf{z})$ with $w_1 = z_1$ and $w_2 = z_2$. This explicit calculation is performed in Appendix C. □

6. CONCLUSIONS

In this chapter we briefly discuss possible improvements, further techniques and future work concerning the results of this thesis.

A great part of our results are limited to the case considered in Problem 1.4 (and its Zariski closed version, Problem 3.11) and in Chapter 5, we observed how this problem is almost trivial when $\varepsilon < 1/2$. Notice that in the range $r = \varepsilon n$, one obtains $\omega = 2(r+1) - n = 2(\varepsilon n + 1) - n = (2\varepsilon - 1)n + 2$; therefore if $\varepsilon > 1/2$, the value ω increases as n increases; this makes difficult to use the equations that we obtain in Section 4.2, because it would not be possible to apply the propagation result of Proposition 4.11 or even a modified ad hoc version as we did in Theorem 5.9. In order to use our method in a more ample range, one needs to develop techniques to evaluate the equations for large ω. There are several possible paths that one can follow. One can investigate techniques to evaluate these equations on matrices of a particular form, such as Vandermonde matrices, whose minors can be expressed in terms of Schur functions, and known results on symmetric functions can be used to reduce the expressions of the equations to a form easy to evaluate. This approach is similar to the one we followed in Chapter 5 in the case of Cauchy matrices, but it would be necessary to develop a method that does not rely on Proposition 4.11. For a second possible approach recall that in the construction of the maps $\psi_\ell^{(n,r)}$, we took *all* possible derivatives of order ℓ in the chosen subset of diagonal entries, and in that case we could guarantee that the diagonal variables were eliminated; one can hope to achieve the same result by taking fewer derivatives, and so control the degree of the resulting equations. This approach relates the equations that we obtained in

Chapter 4 to the study of *resultants*. Appendix B discusses this approach. Finally, one can certainly improve the results obtained via representation theory and probably one can prove similar results in a more general setting. Indeed the arguments that we used in Section 4.3 rely on non-decreasing properties of multiplicities of certain Specht modules inside tensor products (so-called Kronecker coefficients) and can likely be applied whenever these stability results hold. However, our techniques use the propagation result of Proposition 4.11 and one would need further insights to avoid it.

Another challenge is the problem of explicitness. It is often useful to restrict to particular classes of matrices, as we did in Chapter 5, in order to exploit their symmetries: for instance we saw how the $\mathfrak{S}_n$-invariance of $\mathcal{V}and_n$ and the $\mathfrak{S}_n \times \mathfrak{S}_n$-invariance of $\mathcal{C}au_n$ were useful in reducing the number equations to check. However, if one further restricts to a single explicit matrix there are usually two possibilities: either the chosen matrix is *too symmetric* and the equation we are evaluating vanishes, or the symmetry breaks down and one has to check a much higher number of equations.

Finally, one would like to eventually have results in Valiant's range. A natural continuation of the work of [GHIL16] is the study of the hypersurface cases for $r = 2$ and $r = n - 3$. We found several difficulties in both cases. If $r = n - 3$, then the components of $\mathcal{R}[n, r, s]$ are hypersurfaces when $s = 8$, so every configuration S is concentrated in a 8×8 submatrix; if we were able to generalize the method that we used in Section 4.1 in the case $n = 3, r = 1$, we could in principle determine the equation for each configuration S of 8 entries, by propagating the equation that we have in the case $n = 8, r = 5$. This task is computationally challenging both for the difficulty in applying direct elimination in $\mathbb{C}[\mathrm{Mat}_8]$ and for the high number of different cases. As for the case $n = 2$, we get *new* configurations (namely that are not covered by previous cases) when $n \geq 5$. We can perform direct elimination for $n = 5$ and $n = 6$, but we were not able to discover a pattern that could lead to a general form of the equations. Since the challenge in this case is to understand which are the maximal S such that $J(\sigma_r, L^S)$ is a hypersurface, we recall here Remark 3.8. Techniques to study bases and flats of a matroid can be useful to determine the number of irreducible components of $\mathcal{R}[n, r, s]$ and more in particular to characterize the configurations S that give a hypersurface.

APPENDIX A

LINEAR CIRCUITS

This appendix deals with the basic definition of linear circuit, the complexity measure that is used in [Val77] to study matrix-vector multiplication.

Definition A.1. A *linear circuit* is a directed acyclic graph $\mathcal{L}$, with n sources and m sinks. Edges are labeled with constants. Input vertices are labeled with variables $x_1, \dots, x_n$. The other vertices are labeled as follows: if a vertex of $\mathcal{L}$ has in-degree r with in-going edges $a_1, \dots, a_r$ from vertices labeled $u_1, \dots, u_r$, then we label the vertices by $u = a_1 u_1 + \cdots + a_r u_r$ and we say that the vertex *computes* u.

We say that $\mathcal{L}$ *computes* an $n \times n$ matrix A if its sinks compute the entries of $A\mathbf{x}$.

The *size* of a linear circuit $\mathcal{L}$ is the number of edges of $\mathcal{L}$. The *depth* of a linear circuit $\mathcal{L}$ is the maximum length of a directed path in $\mathcal{L}$.

A linear circuit $\mathcal{L}$ that computes a matrix A encodes an algorithm to compute $\mathbf{x} \mapsto A\mathbf{x}$ and the size of the linear circuit counts the number of arithmetic operations used in the algorithm. See [Lok09] for details on this complexity model. For instance the linear circuit in Figure A.1 represents the standard matrix-vector multiplication algorithm for a 3×2 matrix $A = (a_{ij})$.

Similarly, Figure A.2 represents the standard algorithm for a 3×2 matrix $A = (a_{ij})$ with $a_{22} = 0$, and Figure A.3 represents the algorithm for a 3×2 matrix $A = \mathbf{v} \otimes \mathbf{w}$ of rank 1 (with $\mathbf{v} \in \mathbb{C}^3$ and $\mathbf{w} \in \mathbb{C}^2$).

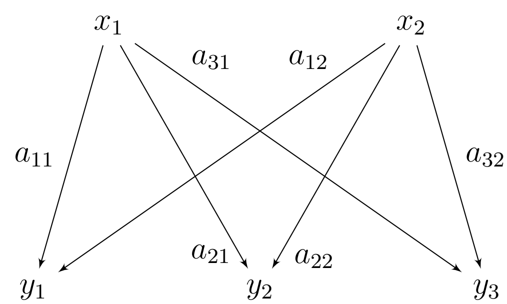


Figure A.1: Standard algorithm for $A \in Mat_{3\times 2}$

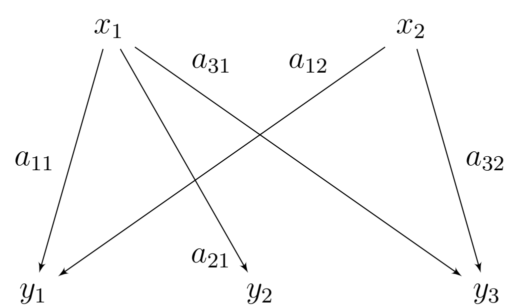


Figure A.2: Standard algorithm for $A \in Mat_{3\times 2}$ with $a_{22} = 0$

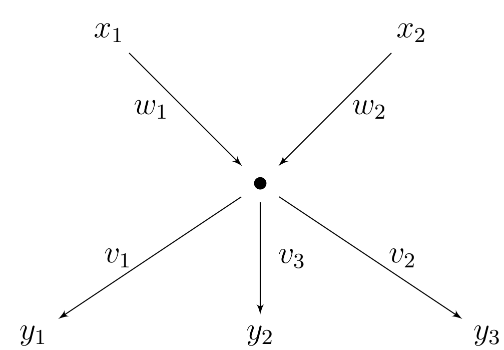


Figure A.3: Standard algorithm for $A = \mathbf{v} \otimes \mathbf{w} \in Mat_{3\times 2}$ of rank 1

Valiant proposed to use matrix rigidity as a measure of the complexity of the linear map associated to A in terms of linear circuits. The following is a more refined version of Theorem 1.2.

Theorem A.2 ([Val77], Thm. 6.1)**.** *Let A_n be a sequence of matrices, with $A_n \in \mathrm{Mat}_n(\mathbb{C})$. Let $\mathcal{L}_n$ be a sequence of in-degree 2 linear circuit of size s_n and depth d_n such that $\mathcal{L}_n$ computes A_n. Fix $t > 1$ and let $\rho_n = \frac{s_n \log(t)}{\log(d_n)}$. Then*

$$\mathrm{Rig}_{\rho_n}(A_n) \leq 2^{O(d/t)} n.$$

The proof is based on a graph-theoretic argument, that we omit. We refer to [Lok09] or to the original source.

APPENDIX B

RESULTANTS

In this section, we propose an approach to the elimination problem in the case where S is diagonal that uses so called *resultants*. Consider a polynomial system in two sets of variables $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ and $k+1$ equations:

$$\begin{aligned} f_0(\mathbf{x}, \mathbf{y}) &= 0 \\ &\vdots \\ f_k(\mathbf{x}, \mathbf{y}) &= 0. \end{aligned} \tag{B.1}$$

Let $I = (f_0, \ldots, f_k) \subseteq \mathbb{C}[x_1, \ldots, x_k, y_1, \ldots, y_m]$. From Theorem 4.1, the elimination ideal $(f_0, \ldots, f_k) \cap \mathbb{C}[y_1, \ldots, y_m]$ cuts out the projection of $V(I)$ modulo the subspace $x_1 = \cdots = x_k = 0$. If the f_j's are sufficiently general, this projection is a hypersurface and its equation is a polynomial in the y_i's that is called *resultant* of $f_0, \ldots, f_k$ with respect to $x_1, \ldots, x_k$.

The theory of resultants is a rich subject that makes use of deep results in algebraic geometry. We refer to [GKZ94] (Ch. 3 and Ch. 4) and [ESW03] for the theory behind resultants and [CLO97] (Ch. 3 and Ch. 7) and [Stu02] for a computational approach.

We are interested in the *multilinear resultant*, that is the resultant of a system of polynomials as in (B.1) that is (affine) multilinear in the $\mathbf{x}$ variables, namely in every monomial of every f_j, each variable x_i appears in degree at most 1. This approach is used in [DSS07] to determine equations for $J(\sigma_r^{(n),sym}, L^S)$, where $\sigma_r^{(n),sym}$ is the variety of $n \times n$ symmetric matrices of rank at most r and S is diagonal.

Consider $p+1$ multilinear polynomials $f_0, \ldots, f_p$ in the variables $t_1, \ldots, t_p$ and $a^j_{\varepsilon_1,\ldots,\varepsilon_p}$ (for $j = 0, \ldots, p, \varepsilon_i \in \{0,1\}$), where $a^j_{\varepsilon_1,\ldots,\varepsilon_p}$ is the coefficient of $t_1^{\varepsilon_1} \cdots t_p^{\varepsilon_p}$ in the polynomial f_j:

$$f_j := \sum_{\varepsilon_1,\ldots,\varepsilon_p \in \{0,1\}} a^j_{\varepsilon_1,\ldots,\varepsilon_p} \cdot t_1^{\varepsilon_1} \cdots t_p^{\varepsilon_p}. \tag{B.2}$$

We regard $I := (f_0, \ldots, f_n)$ as an ideal in the polynomial ring $\mathbb{C}[\mathbf{a}, \mathbf{p}]$.

Theorem B.1 ([DSS07], Thm 19)**.** *The elimination ideal $I \cap \mathbb{C}[\mathbf{a}]$ is principal, generated by an irreducible polynomial $\mathscr{R}(\mathbf{a})$ which is homogeneous of degree $n!$ in the $\mathbf{a}$ variables.*

The polynomial $\mathscr{R}(\mathbf{a})$ is called *n-th multilinear resultant*.

We consider the space of polynomials $\mathbf{E}_\omega^{(n,r)}$ of (4.9). We regard these polynomials as elements of the polynomial ring $\mathbb{C}[x_k^j : j \neq k][x_i^i : i = 1, \ldots, \omega]$; they are (affine) multilinear in the ω diagonal variables appearing in $\mathbf{E}_\omega^{(n,r)}$. Select $\omega + 1$ elements of $\mathbf{E}_\omega^{(n,r)}$, and express them in the form of (B.2) (with $p = \omega$), where now the a's are polynomials in the off diagonal variables x_k^j. The evaluation of the ω-th multlinear resultant at these $\omega+1$ selected polynomials provides a polynomial in the elimination ideal of the ideal generated by $\mathbf{E}_\omega^{(n,r)}$.

Similarly to [DSS07], Thm. 23, we obtain

Theorem B.2. *Let $f_0, \ldots, f_\omega \in \mathbf{E}_\omega^{(n,r)}$ and let $a^j_{\varepsilon_1,\ldots,\varepsilon_\omega}$ be the coefficient of $(x_1^1)^{\varepsilon_1} \cdots (x_\omega^\omega)^{\varepsilon_\omega}$ in f_j. Then $\mathscr{R}(\mathbf{a})$ is an equation for $J(\sigma_r, L^S)$ (possibly zero).*

We believe that this approach can lead to equations for $J(\sigma_r, L^S)$ that have lower degree and are easier to evaluate compared to the ones that we construct in Chapter 4.

APPENDIX C

COMPUTER CALCULATIONS

This appendix deals with explicit computer calculations to prove the base cases of the inductive arguments of Lemma 4.16, Lemma 4.18, Theorem 4.19 and Theorem 5.9. These calculations are performed via the software Macaulay2 ([GS]). We assume some basic knowledge of Macaulay2, and in this appendix, we explain the methods that we used to prove our results; we work over rational number is exact arithmetic; the complete scripts are available at `www.math.tamu.edu/~fulges`.

Specht Modules and Young symmetrizers

A fundamental tool that we used in the proof of the results of Chapter 4 is the Young symmetrizer associated to a standard Young tableau T_λ of shape $\lambda \vdash d$ (for some non negative integer d). Since indices in Macaulay2 start from 0, it is useful to regard the symmetric group $\mathfrak{S}_d$ as acting on the integers $\{0, \ldots, d-1\}$. We represent a Young tableau T_λ as a *list of lists* whose elements are the labels of the boxes of T_λ, listed row by row: for instance

```
T = {{0,1,3}, {2,5}, {4,6}};
```

is the standard Young tableau

$$T_\lambda = \begin{array}{|c|c|c|}\hline 0 & 1 & 3 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2}\end{array}.$$

of shape $\lambda = (3, 2, 2)$.

Given a partition λ, in Chapter 2, we realize the Specht module $[\lambda]$ as the quotient space $V_{\mathcal{T}_\lambda}/K(\lambda)$, the space spanned by Young tableaux modulo the straightening relations. Here, we work with a polynomial ring $\mathcal{V}_d$ whose variables are the Young tableaux (so V_λ is the homogeneous component of degree 1 of $\mathcal{V}_\lambda$ and we consider $\mathcal{K}(\lambda)$, the ideal generated by the straightening relations: therefore $[\lambda]$ is realized as the homogeneous component of degree 1 of the quotient ring $\mathcal{V}_\lambda/\mathcal{K}(\lambda)$. This construction is performed by the file `SpechtModule.m2`. In our construction, the variables of $\mathcal{V}_d$ (namely the Young tableaux) are sorted in such a way that, with the default GRevLex monomial order in Macaulay2, the standard monomials of the quotient $[\lambda]$ are the standard Young tableaux.

Moreover, we explicitly construct the Young symmetrizer associated to a partition λ as explained in Chapter 2. This is performed by the file `YoungSym.m2`, where several *functions* are defined. In particular, the function `YsProj`, takes as input an element f of a tensor product of Specht modules $[\lambda_1] \otimes \cdots \otimes [\lambda_r]$ (expressed as a multilinear polynomial in variables that are in bijections with the standard Young tableaux of shape $\lambda_1, \ldots, \lambda_r$) and a standard Young tableaux T_μ; the output is the projection of f via the Young symmetrizer.

Although one could work with vector spaces and tensor products of them, working with polynomial rings on the set of Young tableaux makes particularly easy to encode the action of the symmetric group, using the `substitute` function of Macaulay2. This will likely affect the performance of the scripts already for moderately small cases but it was useful in the cases concerning this thesis.

The following code proves the base case for the induction argument of Lemma 4.16.

```
1 YD = {2,2}
2 varname0 = T1
3 load("SpechtModule.m2")
4 Sp22copy1 = Sp
5 varname0 = T2
6 load("SpechtModule.m2")
7 Sp22copy2 = Sp
8 Sp22ot2 = Sp22copy1**Sp22copy2;
9 f = T1_{{0,1},{2,3}}*T2_{{0,2},{1,3}} -
10     T1_{{0,2},{1,3}}*T2_{{0,1},{2,3}}
11 load("YoungSym.m2")
12 P = {{0},{1},{2},{3}};
13 YsProj(f,{T1,T2},P)
```

Lines from `1` to `7` define a graded algebra `Sp22ot2` such that its component of multidegree $(1,1)$ is isomorphic to the $\mathfrak{S}_4$-module $[2,2]^{\otimes 2}$. The element `f` at lines `8` and `9` is an element (the unique up to scale) of $\bigwedge^2 [2,2] \subseteq [2,2]^{\otimes 2}$. Line `12` computes the projection of f on $[1^4]$; since it is nonzero, we conclude that $\bigwedge^2 [2,2] \simeq [1^4]$. This proves the base case of the inductive argument of Lemma 4.16.

The same code, with the following changes, proves the base case for the inductive argument of Lemma 4.18

```
1 YD = {3,1,1,1}

9 f = T1_{{0,4,5},{1},{2},{3}}*T2_{{0,3,5},{1},{2},{4}} -
10     T1_{{0,3,5},{1},{2},{4}}*T2_{{0,4,5},{1},{2},{3}}
```

```
P = {{0,1,2,5},{3},{4}};
```

Performing iterated determinants

The induction argument of 4.19 relies on the fact that the invariant contained in $[4,1,1] \otimes \bigwedge^2[3,1,1,1]$ is not in the kernel of the projection $\psi_1 : \bigwedge^3 \mathbf{E}_1^{(8,4)} \to \mathbf{E}_0^{(8,4)}$. This fact is proved by constructing the invariant directly from the minors spanning $\mathbf{E}_2^{(8,4)}$. One can easily check that $[4,1,1] \otimes \bigwedge^2[3,1,1,1]$ only contains one $\mathfrak{S}_6$-invariant up to scale. Because of the way Macaulay2 encodes permutations, we prefer the action of a symmetric group $\mathfrak{S}_6$ that permutes the first 6 rows and columns of an 8×8 matrix (whereas for the discussion of Chapter 4, it was clearer to present the construction by permuting the last $2m$ rows and columns of a matrix).

We work in a polynomial ring whose variables are M_J^I for a suitable set of I, J that appears in the construction. Here `YsProj` is not the same function as in the previous part; it is a refined version that allows us to apply symmetrizations to polynomials in the variables M_J^I. Moreover, applying ψ_1 is computationally too heavy and we defined an alternative function `YsProjMat` that performs the symmetrization that would be needed to apply the Young Symmetrizer on the result of ψ_1 directly on the matrix, working with polynomials of lower degree.

We omit part of the code, that is used to define the following functions:

- `randomOpts()` has no input and its output is a *list* of *options* of the form `M_(I,J) => q_(I,J)`, where q_J^I is the value of the minor M_J^I on a fixed random matrix;

- `YSproj(f,T)` takes as input a polynomial f in the M_J^I's and a tableau T; its output is the projection of f via the Young symmetrizer associated to T;

- `YSprojMat(A,T)` takes as input a matrix A of polynomials in the M_J^I's and a tableau T; its output is a list of matrices, with the property that the sum of their determinants is the projection of $\det(A)$ via the Young symmetrizer of T;

- `Diff(p,f)` takes as input an index $p = 6, 7$ and a polynomial f in the minors M_J^I; its output is the derivative of the polynomial f with respect of the entry x_p^p, expressed in terms of minors.

The following code proves the base case of the inductive argument of Theorem 4.19:

```
T'222_1 = {{0,1},{2,3},{4,5}}
T'222_2 = {{0,2},{1,3},{4,5}}
T'222_3 = {{0,2},{1,4},{3,5}}
gen'222_1 = YSproj(MM_({0,2,4,6,7},{1,3,5,6,7}), T'222_1);
gen'222_2 = YSproj(MM_({0,3,4,6,7},{1,2,5,6,7}), T'222_2);
gen'222_3 = YSproj(MM_({0,3,4,6,7},{1,2,5,6,7}), T'222_3);
w2'222_1 = det(matrix{{gen'222_1,gen'222_2},
        {Diff(6,Diff(7,gen'222_1)),Diff(6,Diff(7,gen'222_2))}});
w2'222_2 = det(matrix{{gen'222_1,gen'222_3},
        {Diff(6,Diff(7,gen'222_1)),Diff(6,Diff(7,gen'222_3))}});
T'21111_1 = {{0,1},{2},{3},{4},{5}};
T'21111_2 = {{0,3},{1},{2},{4},{5}};
gen'21111_1 = YSproj(MM_({0,2,4,6,7},{1,3,5,6,7}), T'21111_1);
gen'21111_2 = YSproj(MM_({0,1,4,6,7},{2,3,5,6,7}), T'21111_2);
w2'21111 = det(matrix{{gen'21111_1,gen'21111_2},
                    {Diff(6,Diff(7,gen'21111_1)),Diff(6,Diff(7,gen'21111_2))}});
```

```
17 mat = matrix {{w2'21111 ,w2'222_1,w2'222_2},
18          {Diff(6,w2'21111), Diff(6,w2'222_1),Diff(6,w2'222_2)},
19          {Diff(7,w2'21111), Diff(7,w2'222_1),Diff(7,w2'222_2)}};
20 T'6 = {{0,1,2,3,4,5}};
21 mat'6 = YSprojMat(mat,T'6);
22 opts = randomOpts();
23 detval = apply(mat'6, mm -> (det(sub(mm,opts))));
24 result = sum(detval)
```

At lines `4,5,6`, we construct elements of $[2,2,2]$ by projecting elements of $\mathbf{E}_2^{(8,4}$ via Young symmetrizers of three standard Young tableaux of shape $[2,2,2]$; call these three elements f_1, f_2, f_3. At lines `7` and `9`, we compute $g_1 = \psi_2(f_1 \wedge f_2)$ and $g_2 = \psi_2(f_1 \wedge f_3)$. Since $\bigwedge^2[2,2,2] = [3,1,1,1]$ is irreducible, g_1, g_2 are already elements of $[3,1,1,1]$ (there is no need of projecting them via a Young symmetrizer). Similarly, at lines `13` and `14`, we construct elements of $[2,1^4] \subseteq \mathbf{E}_0^{(8,4}$, and at line `15`, we compute their image via ψ_2, say g_3; since $\bigwedge^2[2,1,1,1,1] = [4,1,1]$ is irreducible, again there is no need to apply a projection to obtain an element of $[4,1,1]$. At line `17`, we define the matrix whose determinant is $\psi(g_1, g_2, g_3)$. Computing this determinant symbolically is computationally heavy; therefore we apply a modified version of the Young symmetrizer projection onto the invariant space, that acts on the matrix. Lines `22-24` evaluate the image at a random matrix. The `result` value is usually nonzero and this shows that the image of the projection to the invariant space is nonzero. This proves that the image of ψ_1 contains a $\mathfrak{S}_6$-invariant.

Calculation with Cauchy matrices

Finally, to complete the proof of Theorem 5.9, we need to prove that the determinant (5.1) does not vanish on a generic matrix of $Cau_6(\mathbf{w}, \mathbf{z})$ with $w_1 = z_1$ and $w_2 = z_2$. This is performed by the following code, that does not need further explanation.

```
U = (K,L) -> ((product(K, k-> (w_k - w_1)*(w_k - w_2))*
               product(L, l-> (z_l - z_1)*(z_l - z_2)))/
                  (product(K, k-> (w_k + z_1)*(w_k + z_2))*
                   product(L, l-> (w_1 + z_l)*(w_2 + z_l))));
V = (e,K,L) -> ((product(K, k-> (w_k - w_e))*
                 product(L, l-> (z_l - z_e)))/
                    (product(K, k-> (w_k + z_e))*
                     product(L, l-> (w_e + z_l))));
difU = (K,L) -> (U(K,L) - U(L,K));
difV = (e,K,L) -> (V(e,K,L) - V(e,L,K));
for i from 1 to 6 do (w_i = random(QQ));
z_1 = w_1;
z_2 = w_2;
for i from 3 to 6 do (z_i = random(QQ));
m =matrix{{difU({3,4},{5,6}), difU({3,5},{4,6}) , difU({3,6},{4,5})},
    {difV(2,{3,4},{5,6}), difV(2,{3,5},{4,6}) , difV(2,{3,6},{4,5})},
    {difV(1,{3,4},{5,6}), difV(1,{3,5},{4,6}) , difV(1,{3,6},{4,5})}}
det m
```

Usually the result of line `a` is nonzero, that proves the base case of the induction argument of Theorem 5.9.

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