

# THE CONVEXITY BOUND FOR RANKIN-SELBERG $L$ -FUNCTIONS

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## 1. INTRODUCTION

The Phragmen-Lindelöf convexity principle allows one to produce a bound on an  $L$ -function inside the critical strip by interpolating bounds outside but near the right and left edges of the critical strip. The functional equation allows one to produce a bound on the left edge using a bound on the right edge. However, for Dirichlet series with potentially large coefficients, it is not trivial to bound the  $L$ -function on the right edge.

For example, consider a Maass form  $F$  on  $SL_n(\mathbb{Z})$  that is an eigenfunction for the full Hecke ring. By an integral representation involving Eisenstein series, one can construct the Rankin-Selberg  $L$ -function  $L(F \times \overline{F}, s)$  (see (2.1) below) which is entire except for a simple pole at  $s = 1$ , satisfies a functional equation under  $s \rightarrow 1 - s$ , and has a degree  $n^2$  Euler product. Except for the notable case of  $n = 2$ , [R], this  $L$ -function is not known to correspond to an automorphic form. The analytic properties of  $L(F \times \overline{F}, s)$  are crucial in understanding the size of the Hecke eigenvalues of  $F$ .

Iwaniec [I] considered the case  $n = 2$  and showed  $S(x) := \sum_{n \leq x} |\lambda_j(n)|^2 \ll t_j^\varepsilon x^{1+\varepsilon}$ , where  $\lambda_j(n)$  are the Hecke eigenvalues of a Maass form  $u_j$  with spectral parameter  $t_j$ , and the implied constant does not depend on  $t_j$ . This strong bound is in spite of the fact that we do not know the Ramanujan conjecture for Maass forms, and indeed the current best bound is  $|\lambda_j(n)| \ll n^{7/64+\varepsilon}$  [KS]. The wonderful idea in Iwaniec's method is to deduce a nonlinear recurrence for  $S(x)$  in the form  $S(x)^2 \ll x^\varepsilon \sum_{d \leq x} S(x^2/d^2)$ , with an implied constant independent of  $t_j$ . For  $x$  large enough compared to  $t_j$  one can use analytic properties of the Rankin-Selberg  $L$ -function to deduce the desired bound, and the recurrence then transfers this bound to smaller values of  $x$ . The multiplicativity of the Hecke eigenvalues plays a vital role in the method. Iwaniec's bound on  $S(x)$  then shows that  $L(u_j \times u_j, s)$  satisfies the convexity bound which we state in a slightly nontraditional way as  $L(u_j \times u_j, 1 + \delta + it) \ll_{\delta, \varepsilon} t_j^\varepsilon$ .

Molteni [M] generalized Iwaniec's method and considered the more general problem of bounding higher degree  $L$ -functions. He succeeded at giving the convexity bound for the standard  $L$ -function on  $GL_n$  but not for the Rankin-Selberg  $L$ -function except with an additional hypothesis.

Brumley [B] later obtained convexity for  $GL_n \times GL_n$  with  $n \leq 4$ . Recently, Li [L] obtained the convexity bound on  $GL_n \times GL_n$  for any  $n$ ; his lovely idea is to study the logarithmic derivative of the  $L$ -function, and notice that for purposes of obtaining an upper bound the zeros are harmless. This idea was initially developed by Soundararajan [S] inside the critical strip where it is conditional on GRH. Li's method is unconditional since the setting is to the right of the critical strip where the Euler product converges absolutely.

For conceptual reasons, it is still interesting to understand if Iwaniec's method can be generalized to  $GL_n \times GL_n$ , or if there is some barrier. Here we in fact confirm that the method generalizes.

**Theorem 1.1.** *Fix  $n \geq 2$ . Let  $F$  be a Maass form on  $SL_n(\mathbb{Z})$  that is an eigenfunction for the full Hecke ring with Laplace eigenvalue  $\lambda_F(\Delta)$ . Then for any  $\varepsilon, \delta > 0$ ,*

$$(1.1) \quad L(F \times \overline{F}, 1 + \delta + it) \ll_{\delta, \varepsilon} \lambda_F(\Delta)^\varepsilon,$$

*holds with an implied constant independent of  $F$ ; that is, the convexity bound holds.*

In fact, we show an independently interesting bound which implies Theorem 1.1.

**Theorem 1.2.** *For any  $\varepsilon > 0$  there exists a constant  $C(\varepsilon)$  such that for any  $F$  as in Theorem 1.1 and any  $s > 1 + \varepsilon$ , the following holds*

$$(1.2) \quad L(F \times \overline{F}, s)^2 \leq C(\varepsilon)L(F \times \overline{F}, s - \varepsilon)$$

Note that  $L(F \times \overline{F}, s)$  is nonnegative for all  $s > 1$  since its Dirichlet series has nonnegative coefficients. Hence  $|L(F \times \overline{F}, \sigma + it)| \leq L(F \times \overline{F}, \sigma)$  for  $\sigma > 1$  and  $t \in \mathbb{R}$ . This self-referential bound could be considered the Dirichlet series analog of Iwaniec's recursive bound on  $S(x)$ . However, we take the opportunity to point out a seemingly innocuous modification of  $S(x)$  which simplifies Iwaniec's argument. Let  $S^*(x) = \sum_{m^2 n \leq x} |\lambda_j(n)|^2$ . One can see that  $S^*(x)$  is very natural since it is the summatory function for the Dirichlet series coefficients of  $L(u_j \times u_j, s)$ . Clearly  $S(x) \leq S^*(x)$ . Iwaniec's recursive argument gives  $S^*(x)^2 \leq C(\varepsilon)x^\varepsilon S^*(x^2)$  which avoids the extra summation variable meaning that these coefficients have a nice closure property under multiplication. This closure property holds for higher degree also which turns out to be a consequence of well-known properties of Schur polynomials (see (2.10) below). We observe that the coefficients of  $L(F, s)$  itself do not have this nice closure property.

To deduce Theorem 1.1 from Theorem 1.2, one begins with a bound of the form  $L(F \times \overline{F}, 1 + \delta) \ll_\delta \lambda_F(\Delta)^A$  for some large  $A$  independent of  $F$ . Then applying Theorem 1.2 with  $\varepsilon = \delta/2$  effectively replaces  $A$  by  $A/2$ . By repeating this procedure we can make  $A$  arbitrarily small. We also mention that it is easy to deduce the convexity bound on  $L(F_1 \times F_2, s)$  where  $F_i$  is a Maass form on  $SL_{n_i}(\mathbb{Z})$  for  $n_i = 1, 2$  by Cauchy's inequality.

**Corollary 1.3.** *Let notation be as above and write  $L(F \times \overline{F}, s) = \sum_{m=1}^\infty a_F(m)m^{-s}$ . Then*

$$(1.3) \quad \sum_{m \leq x} a_F(m) \ll x^{1+\varepsilon} \lambda_F(\Delta)^\varepsilon,$$

*with an implied constant depending on  $\varepsilon > 0$  only.*

This follows from Theorem 1.1 by the standard contour integral method.

## 2. BACKGROUND

By Definition 12.1.2 of [G], say, we have for  $s > 1$

$$(2.1) \quad L(F \times \overline{F}, s) = \zeta(ns) \sum_{m_1, \dots, m_{n-1}} \frac{|A(m_1, \dots, m_{n-1})|^2}{(m_1^{n-1} m_2^{n-2} \dots m_{n-1})^s},$$

where  $A(m_1, \dots, m_{n-1})$  are the Fourier coefficients of  $F$ , normalized so that  $A(1, \dots, 1) = 1$ . The absolute convergence for  $\text{Re}(s) > 1$  is ensured by Landau's lemma and the fact that the Rankin-Selberg  $L$ -function has an analytic continuation to  $\mathbb{C}$  with a simple pole at  $s = 1$

only. Since the Hecke eigenvalues are multiplicative in the joint sense, we have by general principles of multiplicative functions that

$$(2.2) \quad L(F \times \bar{F}, s) = \prod_p \sum_{k, k_1, \dots, k_{n-1}=0}^{\infty} \frac{|A(p^{k_1}, \dots, p^{k_{n-1}})|^2}{(p^s)^{nk+k_1(n-1)+k_2(n-2)+\dots+k_{n-1}}}.$$

We find it convenient to work with the Schur polynomial representation for each Euler factor. Goldfeld ([G] p.367) remarks that

$$(2.3) \quad A(p^{k_1}, p^{k_2}, \dots, p^{k_{n-1}}) = S_{k_1, \dots, k_{n-1}}(\alpha_{p,1}, \dots, \alpha_{p,n}),$$

where  $S_{k_1, \dots, k_{n-1}}(x_1, \dots, x_n)$  is the Schur polynomial defined as follows

$$(2.4) \quad S_{k_1, \dots, k_{n-1}}(x_1, \dots, x_n) = \frac{\begin{vmatrix} x_1^{k_1+k_2+\dots+k_{n-1}+n-1} & \dots & x_n^{k_1+k_2+\dots+k_{n-1}+n-1} \\ x_1^{k_1+k_2+\dots+k_{n-2}+n-2} & \dots & x_n^{k_1+k_2+\dots+k_{n-2}+n-2} \\ \vdots & \vdots & \vdots \\ x_1^{k_1+1} & \dots & x_n^{k_1+1} \\ 1 & \dots & 1 \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & \dots & x_n^{n-2} \\ \vdots & \vdots & \vdots \\ x_1 & \dots & x_n \\ 1 & \dots & 1 \end{vmatrix}},$$

and  $\alpha_{p,1}, \dots, \alpha_{p,n}$  are the Satake parameters defined by the Euler product formula

$$(2.5) \quad L(F, s) = \sum_{m=1}^{\infty} A(m, 1, \dots, 1) m^{-s} = \prod_p \prod_{i=1}^n (1 - \alpha_{p,i} p^{-s})^{-1}.$$

To connect more easily to the literature on Schur polynomials we prefer to express things in terms of Schur functions indexed by partitions (into at most  $n$  parts)  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ . Precisely, let

$$(2.6) \quad s_{\lambda}(x_1, \dots, x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|},$$

where the notation indicates the determinant of the matrix with given  $i, j$  entry. We have

**Lemma 2.1.** *Let  $F$  be a Hecke-Maass form for  $SL_n(\mathbb{Z})$ , and suppose  $s > 1$ . Then with  $x_p = p^{-s/2}$ , we have*

$$(2.7) \quad L(F \times \bar{F}, s) = \prod_p \sum_{\lambda} |s_{\lambda}(x_p \alpha_{p,1}, \dots, x_p \alpha_{p,n})|^2.$$

*Proof.* Define  $\lambda = (\lambda_1, \dots, \lambda_n)$  via

$$\begin{aligned} \lambda_1 &= k_1 + k_2 + \dots + k_{n-1} + k \\ \lambda_2 &= k_1 + k_2 + \dots + k_{n-2} + k \\ &\vdots \\ \lambda_{n-1} &= k_1 + k \\ \lambda_n &= k. \end{aligned}$$

As  $k_1, \dots, k_{n-1}, k$  run over all nonnegative integers,  $\lambda$  runs over all partitions of integers with at most  $n$  parts. Note that

$$(2.8) \quad nk + k_1(n-1) + k_2(n-2) + \dots + k_{n-1} = \lambda_1 + \lambda_2 + \dots + \lambda_n,$$

and so using standard properties of determinants,

$$(2.9) \quad \frac{S_{k_1, \dots, k_{n-1}}(\alpha_{p,1}, \dots, \alpha_{p,n})}{(p^{s/2})^{nk+k_1(n-1)+\dots+k_{n-1}}} = (\alpha_{p,1} \dots \alpha_{p,n})^{-\lambda_n} s_\lambda(p^{-s/2}\alpha_1, \dots, p^{-s/2}\alpha_n).$$

With (2.2) and (2.3), this proves the result.  $\square$

The Schur polynomials form a natural basis for the vector space of degree  $d$  homogeneous polynomials in  $n$  variables. In particular, we have

$$(2.10) \quad s_\lambda s_\mu = \sum_{\nu} N_{\lambda\mu\nu} s_\nu,$$

where  $\nu$  runs over certain partitions with  $|\nu| = |\lambda| + |\mu|$  (everything with at most  $n$  parts), and  $N_{\lambda\mu\nu}$  is a nonnegative integer. The well-known Littlewood-Richardson rule gives a combinatorial interpretation for  $N_{\lambda\mu\nu}$  yet this description is rather involved.

**Lemma 2.2.** *For  $n$  fixed, we have that  $\sum_{\nu} N_{\lambda\mu\nu}$  is bounded by a polynomial in  $\lambda$  and  $\mu$  (meaning, bounded by a polynomial in the components  $\lambda_i$  and  $\mu_j$ ). In particular, each  $N_{\lambda\mu\nu}$  is polynomially bounded by  $\lambda$  and  $\mu$  and so is the number of  $\nu$  on the right hand side of (2.10) such that  $N_{\lambda\mu\nu}$  is nonzero,*

*Proof.* By A.30(ii) of [FH] we have

$$(2.11) \quad s_\lambda(1, 1, \dots, 1) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

In particular,  $s_\nu(1, \dots, 1) \geq 1$  and so evaluating both sides of (2.10) at  $(1, \dots, 1)$ , we have  $\sum_{\nu} N_{\lambda\mu\nu} \leq s_\lambda(1, \dots, 1) s_\mu(1, \dots, 1)$ .  $\square$

### 3. PROOF OF THEOREM 1.2

Suppose that  $s > 1 + \varepsilon$ . Then by Lemma 2.1, we have

$$(3.1) \quad L(F \times \overline{F}, s)^2 = \prod_p \sum_{\lambda} \sum_{\mu} |s_\lambda(x_p \alpha_{p,1}, \dots, x_p \alpha_{p,n}) s_\mu(x_p \alpha_{p,1}, \dots, x_p \alpha_{p,n})|^2.$$

By (2.10), we have

$$(3.2) \quad L(F \times \overline{F}, s)^2 = \prod_p \sum_{\lambda} \sum_{\mu} \left| \sum_{\nu} N_{\lambda\mu\nu} s_\nu(x_p \alpha_{p,1}, \dots, x_p \alpha_{p,n}) \right|^2.$$

By Cauchy's inequality and positivity, we have

$$(3.3) \quad L(F \times \overline{F}, s)^2 \leq \prod_p \sum_{\lambda} \sum_{\mu} \sum_{\nu'} N_{\lambda\mu\nu'}^2 \sum_{\nu: |\nu|=|\lambda|+|\mu|} |s_\nu(x_p \alpha_{p,1}, \dots, x_p \alpha_{p,n})|^2.$$

Given a partition  $\nu = (\nu_1, \dots, \nu_n)$  with  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_n$ , a crude counting argument shows that the number of partitions  $\lambda, \mu$  each of at most  $n$  parts such that  $|\lambda| + |\mu| = |\nu|$  is bounded by a polynomial in  $\nu$ . This fact, together with Lemma 2.2 gives

$$(3.4) \quad L(F \times \overline{F}, s)^2 \leq \prod_p \sum_{\nu} \text{Poly}(\nu) |s_\nu(x_p \alpha_{p,1}, \dots, x_p \alpha_{p,n})|^2,$$

where  $Poly(\nu)$  is a polynomial in  $\nu_1, \dots, \nu_n$ . One also observes directly that if  $\nu = (0, \dots, 0)$  then  $Poly(\nu) = 1$ . Using (2.9) and changing variables back to  $k_1, \dots, k_n$ , we see that

$$(3.5) \quad L(F \times \overline{F}, s)^2 \leq \prod_p \sum_{k, k_1, \dots, k_{n-1}=0}^{\infty} Poly(k, k_1, \dots, k_{n-1}) \frac{|A(p^{k_1}, \dots, p^{k_{n-1}})|^2}{(p^s)^{nk+k_1(n-1)+k_2(n-2)+\dots+k_{n-1}}}.$$

Now by borrowing  $\varepsilon$  from  $s$ , we have

$$(3.6) \quad L(F \times \overline{F}, s)^2 \leq \prod_p \sum_{k, k_1, \dots, k_{n-1}=0}^{\infty} \frac{Poly(k, k_1, \dots, k_{n-1})}{(p^\varepsilon)^{nk+k_1(n-1)+k_2(n-2)+\dots+k_{n-1}}} \frac{|A(p^{k_1}, \dots, p^{k_{n-1}})|^2}{(p^{s-\varepsilon})^{nk+k_1(n-1)+k_2(n-2)+\dots+k_{n-1}}}.$$

For each  $p$ , let  $L_p = (l, l_1, \dots, l_{n-1})$  denote the index for which

$$(3.7) \quad r(L_p) := \frac{Poly(l, l_1, \dots, l_{n-1})}{(p^\varepsilon)^{nl+l_1(n-1)+l_2(n-2)+\dots+l_{n-1}}}$$

is maximized. Since the denominator has exponential growth, there are only finitely many  $p$  such that this index is not  $(0, \dots, 0)$  in which case this ratio is 1 (it is vital this polynomial takes the value 1 but of course this is easily achieved). Then a standard divisor function bound shows that  $\prod_p r(L_p) \leq C(\varepsilon)$  independently of  $F$ . Bounding the polynomial part by  $r(L_p)$  and summing over the  $k$ 's finishes the proof of Theorem 1.2.

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