MOMENTS OF THE DISTRIBUTION FUNCTION OF A LACUNARY FOURIER SERIES AND THE GAUSSIAN NORMAL DISTRIBUTION

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June 1999

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I. Suppose that we are given a sequence of positive integers $n_1 < n_2 < n_3 < \ldots$ such that $n_{j+1} / n_j > q > 1$ for all j. We call such a sequence lacunary.

The problem of studying the value distribution of the partial sums $S_N(x)$ of the Fourier series

$$\sum_{k=1}^{\infty} a_k \cos(2\pi n_k x) + b_k \sin(2\pi n_k x)$$

is discussed in [K], [Z1, section 5.6], and [Z2, section 16.5] under a variety of conditions beginning with the basic hypothesis that

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \infty.$$

Perhaps the most important result one finds is the following. Let

$$A_N = \frac{1}{2}\sum_{k=1}^{N}(a_k^2 + b_k^2),\ S_N(x) = \sum_{k=1}^{N} a_k \cos(2\pi n_k x) + b_k \sin(2\pi n_k x).$$

Then, for any $-\infty < \alpha < \beta < \infty$,

$$\lim_{N\to\infty} m\{x \in [0,1] : \alpha \le \frac{S_N(x)}{\sqrt{A_N}} \le \beta\} = \frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{-v^2/2} dv. \tag{1}$$

This result holds, for instance, any time the a_k and b_k are uniformly bounded. Cf [Z2, section 6.5]. We call (1) an asymptotic Gaussian law. The customary method-of-proof involves the use of characteristic functions in the sense of probability theory.

It is known, however, that the validity of an asymptotic Gaussian law does *not* automatically carry with it convergence of moments of order three or more.

In this report we wish to formulate a theorem which states that, under a natural set of conditions on the coefficients a_k and b_k, the moments *do*, in fact, converge as they should. Technically, what we need to check is that

$$\int_0^1 \left(\frac{S_N(x)}{\sqrt{A_N}}\right)^k dx \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^k e^{-u^2/2} du,$$

i.e.,

$$\int_0^1 \left(\frac{S_N(x)}{\sqrt{A_N}}\right)^k dx \to \begin{cases} 0, & k \text{ odd} \\ 2^{-k/2}\dfrac{k!}{(k/2)!}, & k \text{ even}. \end{cases} \tag{2}$$

A result hinting at some conditions for this is found in [K, pp. 43-44]. When (2) does hold, equation (1) is an automatic consequence; cf. section IV below.

It will be convenient to state our result in two steps. The first is as follows.

THEOREM 1. Let $\{n_j\}$ be any lacunary sequence. Given any sequence of complex numbers $c_j = a_j - ib_j$ such that

(3) $$|c_j| \leq M_0$$

and

(4) $$M_1 \leq \frac{\sum_{j=1}^{N} |c_j|^2}{N} \leq M_2$$

for certain positive M_0, M_1, M_2. Let

$$T_N(x) = \sum_{j=1}^{N} c_j \exp(2\pi i n_j x) \ , \ \ A_N = \frac{1}{2}\sum_{j=1}^{N} |c_j|^2 .$$

Then:

$$\lim_{N\to\infty} \int_0^1 \left(\frac{T_N(x)}{\sqrt{A_N}}\right)^A \left(\frac{\overline{T_N(x)}}{\sqrt{A_N}}\right)^B dx = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \iint_{\mathbb{R}^2} (u+iv)^A (u-iv)^B e^{-\frac{1}{2}(u^2+v^2)} du dv$$

$$= \begin{cases} 0, \ if \ \ A \neq B \\ 2^A A!, \ if \ \ A = B \end{cases}$$

for any choice of nonnegative integers A and B. [We note that condition (4) is satisfied, for instance, anytime $\sum_{j=1}^{N} |c_j|^2 \sim CN$ holds for some positive C as $N \to \infty$.]

The uniformity inherent in (4) enables one to give a proof of theorem 1 using only relatively simple counting arguments. One can go further, however. Namely:

THEOREM 1bis. In theorem 1, condition (4) can be replaced by the much weaker hypothesis that

(4') $$\sum_{j=1}^{\infty} |c_j|^2 = \infty.$$

The resulting limit formulae will be the same as before.

COROLLARY 1. Under either set of hypotheses,

$$\int_0^1 \left(\frac{S_N(x)}{\sqrt{A_N}}\right)^k dx \to \begin{cases} 0, \ k \ odd \\ 2^{-k/2} \dfrac{k!}{(k/2)!}, \ k \ even \end{cases}$$

COROLLARY 2. We also have:

$$\lim_{N\to\infty} m\left\{x \in [0,1] : \alpha \leq \frac{S_N(x)}{\sqrt{A_N}} \leq \beta, \ \gamma \leq \frac{S_N^*(x)}{\sqrt{A_N}} \leq \delta\right\} = \left(\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-u^2/2} du\right)\left(\frac{1}{\sqrt{2\pi}}\int_\gamma^\delta e^{-v^2/2} dv\right)$$

for any $\alpha < \beta, \gamma < \delta$. Here

$$S_N(x) = \text{Re} \ \ T_N(x) = \sum_{j=1}^{N} (a_j \cos(2\pi n_j x) + b_j \sin(2\pi n_j x))$$

$$S_N^*(x) = \text{Im} \ \ T_N(x) = \sum_{j=1}^{N} (-b_j \cos(2\pi n_j x) + a_j \sin(2\pi n_j x)).$$

Corollary 2 states that $S_N(x)$ and $S^*_N(x)$ are asymptotically statistically independent. This corollary represents a natural strengthening of the results mentioned in [Z2] and [K]. See also [SZ, assertion (vi)], where the method used is one of characteristic functions.

II. In this section we prove theorem 1. The proof requires a series of elementary lemmas. To save space, we will occasionally suppress minor details in the proofs of the lemmas. We also use some special notation. The symbol ((h)) will occasionally be used to signify a real number in the interval [-h,h]. The notation B(q) will signify a positive constant depending solely on q. Its value can change from location to location; i.e. we will not insert a subscript each time a new constant is needed. Similarly for B(q,j), B(q,j,k), etc.

For our given q, we set

$$M = \frac{q+3}{q-1}$$

and observe that

$$1 < M < \infty;$$

(5)
$$1 < \frac{1+\frac{1}{M}}{1-\frac{1}{M}} < q.$$

LEMMA 1. For all $a \geq 1$, $k \geq a+1$,

$$\frac{n_{k-a}}{n_k} < q^{-a}.$$

Proof.
Obvious. ■

LEMMA 2. Let $j \geq 1$ and $n \in \mathbb{Z}$. The number of solutions of

$$\pm n_{k_1} \pm n_{k_2} \pm \ldots \pm n_{k_j} = n$$

with distinct indices $1 \leq k_i \leq N$ is bounded by

$$B(q,j) N^{j/2-\delta(j)}$$

where

$$\delta(j) = \begin{cases} 1/2, & j \equiv 1 \mod 2 \\ 0, & j \equiv 0 \mod 2. \end{cases}$$

The estimate is uniform in n. typo, should be 1, not 0.

Proof.
In proving this result, it clearly suffices to look at just the case where the k_i are placed in descending order and the n_{k_1} term appears with a plus sign. We propose to use induction.

The case j=1 is trivial.

The case j=2 leads to $n_{k_1} \pm n_{k_2} = n$. Consider first those solutions with $n_{k_2} / n_{k_1} < 1/M$. In this case, we have

$$n = n_{k_1} \pm n_{k_2} = n_{k_1}[1 + ((1/M))].$$

The number of admissible n_{k_1} is thus at most one. See equation (5). The total number of solutions of this type is thus at most 2 (one for each choice of the plus or minus sign). It remains to treat the solutions with $n_{k_2} / n_{k_1} \geq 1/M$. We write $k_2 = k_1 - a$ and note that $a \leq B(q)$ by lemma 1. For fixed a, the number of solutions is readily seen to be ≤ B(q). Indeed:

$$n = n_{k_1} \pm n_{k_1 - a} = n_{k_1}[1 + ((q^{-a}))] = n_{k_1}[1 + ((q^{-1}))].$$

Collectively, then, for j=2, the number of solutions is ≤ B(q). Compare [Z1, pp. 203-204].

At this point, take j ≥ 3 and assume that the lemma holds for all earlier "j." By lemma 1, there is a number y(q,j) such that, for each choice of n_{k_2}, there are at most y(q,j) choices of n_{k_1} such that $n_{k_2}/n_{k_1} \geq (Mj - M)^{-1}$. Therefore, for each choice of n_{k_2} there are at most $B(q, j-2)y(q,j)N^{(j-2)/2-\delta(j)}$ solutions to $\pm n_{k_1} \pm n_{k_2} \pm \ldots \pm n_{k_j} = n$ with $n_{k_2}/n_{k_1} \geq (Mj - M)^{-1}$, since we can subtract the n_{k_1} and n_{k_2} terms from both sides of the equation and use the induction hypothesis. The number of choices for n_{k_2} is at most N. It follows that there are at most $B(q, j)N^{j/2-\delta(j)}$ solutions of $\pm n_{k_1} \pm n_{k_2} \pm \ldots \pm n_{k_j} = n$ with $n_{k_2}/n_{k_1} \geq (Mj - M)^{-1}$.

If $n_{k_2}/n_{k_1} < (Mj - M)^{-1}$, then $n_{k_1} > n_{k_2} + \ldots + n_{k_j}$ and

$$n = n_{k_1}[1 + ((1/M))].$$

For given n, the number of admissible n_{k_1} is thus at most one. See (5). Simultaneously, the number of n_{k_2} is at most N. By repeating the earlier subtraction of the n_{k_1} and n_{k_2} terms, we see that there are at most $B(q, j)N^{j/2-\delta(j)}$ solutions of $\pm n_{k_1} \pm n_{k_2} \pm \ldots \pm n_{k_j} = n$ with $n_{k_2}/n_{k_1} < (Mj - M)^{-1}$. All told, then, the collective cardinality is ≤ $B(q, j)N^{j/2-\delta(j)}$.

This completes the induction. ■

<u>LEMMA 3.</u> Let j ≥ 1 and n ∈ $\mathbb{Z}$. Consider representations of n of the form

$$\pm r_1 n_{k_1} \pm r_2 n_{k_2} \pm \ldots \pm r_\mu n_{k_\mu}$$

where the k_i are <u>distinct</u>, $1 \leq k_i \leq N$, and the r_i are positive integers that sum to j. Determine a point of $\mathbb{Z}^j$ by making a concatenation of r_1 $\pm k_1$'s, r_2 $\pm k_2$'s, … , r_μ $\pm k_\mu$'s. The total number of such points that represent n is at most $B(q, j)N^{j/2-\delta(j)}$ uniformly in n.

<u>Proof.</u>

The case j = 1 is trivial.

The case j = 2 corresponds to one of:

(a) $n = \pm 2n_{k_1}$, $\mu = 1$

(b) $n = \pm n_{k_1} \pm n_{k_2}$, $\mu = 2$.

The cardinality in (a) is at most 2. The cardinality in (b) is at most B(q) by lemma 2. The total for j=2 is thus ≤ B(q).

Now let j ≥ 3 and suppose that lemma 3 is proved for all "weights" smaller than j. If the r_i's are all 1, simply use lemma 2. Suppose now that $r_1 \geq 2$, for instance. If $r_1 = j$, the cardinality is at most 2. If $2 \leq r_1 < j$, we apply the induction hypothesis to

$$\pm r_2 n_{k_2} \pm \ldots \pm r_\mu n_{k_\mu} = n \mp r_1 n_{k_1},$$

viewed as weight j - r_1. For *each* choice of n_{k_1}, there are at most

$$B(q, j - r_1)N^{(j-r_1)/2-\delta(j-r_1)}$$

solution points. If $r_1 = 2$, then δ(j - r_1) = δ(j) and the number of solution points is at most

$$B(q, j-2)N^{(j-2)/2-\delta(j)} \cdot N = B(q, j-2)N^{j/2-\delta(j)}.$$

If $r_1 \geq 3$, then the number of solution points is at most

$$B(q, j - r_1)N^{(j-r_1)/2-\delta(j-r_1)} \cdot N \leq B(q, j - r_1)N^{j/2-1}.$$

But j/2 – 1 ≤ j/2 - δ(j). Summing over the various possibilities for r_1 shows that the total cardinality is at most $B(q, j)N^{j/2-\delta(j)}$.

This completes the induction. ∎

DEFINITION. Let A and B be positive integers. Let $n \in \mathbb{Z}$. A solution of

$$n_{k_1} + n_{k_2} + \ldots + n_{k_A} = n_{m_1} + n_{m_2} + \ldots + n_{m_B} + n$$

with $1 \le k_i \le N$, $1 \le m_j \le N$ (repetitions allowed) is said to be irreducible if $n_{k_i} \ne n_{m_j}$ for all i, j. We make an obvious extension to cover the case in which exactly one of A and B is 0.

LEMMA 4. Let $n \in \mathbb{Z}$. Let A and B be distinct nonnegative integers. The number of solutions of

$$n_{k_1} + n_{k_2} + \ldots + n_{k_A} = n_{m_1} + n_{m_2} + \ldots + n_{m_B} + n$$

with $1 \le k_i \le N$, $1 \le m_j \le N$ (repetitions allowed) is uniformly at most

$$B(q, A, B) N^{(A+B)/2 - \delta(A+B)}$$

Proof.
A moment's thought shows that it suffices to prove the foregoing bound for irreducible solutions. One simply "maps" each reducible solution down to an irreducible one of type (A-r, B-r) for some $r \ge 1$ and then works back modulo permutations. The case of irreducible solutions corresponds to representations of n as

$$n = n_{k_1} + n_{k_2} + \ldots + n_{k_A} - n_{m_1} - n_{m_2} - \ldots - n_{m_B}$$

with possible repetitions in the k_i and in the m_j, but $k_i \ne m_j$ for all i, j. This is taken care of by lemma 3 (modulo trivial permutations). ∎

LEMMA 5. Let $n \in \mathbb{Z} \setminus \{0\}$. Let A be a positive integer. The number of solutions of

$$n_{k_1} + n_{k_2} + \ldots + n_{k_A} = n_{m_1} + n_{m_2} + \ldots + n_{m_A} + n$$

with $1 \le k_i \le N$, $1 \le m_j \le N$ (repetitions allowed) is uniformly at most

$$B(q, A) N^{A-1}.$$

Proof.
Mimic the proof of lemma 4. ∎

LEMMA 6. Let A be a positive integer. The number of solutions of

$$n_{k_1} + n_{k_2} + \ldots + n_{k_A} = n_{m_1} + n_{m_2} + \ldots + n_{m_A}$$

with $1 \le k_i \le N$, $1 \le m_j \le N$ (repetitions allowed) which are not reducible to "0 = 0" by cancellation is at most

$$B(q, A) N^{A-1}.$$

Proof.
Paraphrase the proof of lemma 4 once again. ∎

We now exploit orthogonality to evaluate

$$I_{AB} = \int_0^1 T_N(x)^A \overline{T_N(x)}^B dx.$$

This expression is clearly

$$\sum c_{k_1} c_{k_2} .. c_{k_A} \overline{c_{m_1}} ... \overline{c_{m_B}}$$

summed over all (vector-type) solutions of

$$n_{k_1} + n_{k_2} + \ldots + n_{k_A} = n_{m_1} + n_{m_2} + \ldots + n_{m_B}$$

where $1 \le k_i \le N$, $1 \le m_j \le N$ (repetitions allowed).

For $A \ne B$, by (3) and lemma 4, we get

$$|I_{AB}| \le M_0^{A+B} B(q, A, B) N^{(A+B)/2 - \delta(A+B)}.$$

Therefore, by (4), one has

(6) $$\lim_{N\to\infty}\int_0^1\left(\frac{T_N(x)}{\sqrt{A_N}}\right)^A\left(\overline{\frac{T_N(x)}{\sqrt{A_N}}}\right)^B dx=0$$

for $A \neq B$.

To finish proving theorem 1, it remains to handle the cases with A = B. The case A = B = 0 is trivial. The case A = B = 1 follows from Parseval's Formula; i.e.,

$$I_{11}=\sum_{j=1}^{N}\left|c_j\right|^2$$

whereupon $I_{11} / A_N = 2$ for all N (not merely as N tends to infinity).

We can now suppose that $A = B \geq 2$. We need to estimate

$$\sum c_{k_1}c_{k_2}\dots c_{k_A}\overline{c_{m_1}}\dots\overline{c_{m_A}}$$

summed over solutions of

$$n_{k_1}+\dots+n_{k_A}=n_{m_1}+\dots+n_{m_A}$$

with $1 \leq k_i \leq N$, $1 \leq m_j \leq N$ and possible repetitions among the k_i and m_j.

The contribution from those vector-type solutions which are not reducible to 0 = 0 is, by lemma 6, of absolute value no more than

(7) $$M_0^{\;2A}B(q,A)N^{A-1}.$$

The contribution from those vector-type solutions which *are* reducible to 0 = 0 requires some thought, because, for instance, the 2A-tuple (k_1, k_1, … k_1) gives only one vector irrespective of any permutations.

The 0 = 0 contribution clearly reduces to

(*) $$\sum\left|c_{k_1}\right|^2\dots\left|c_{k_A}\right|^2,$$

summed over all vector-type solutions of

$$n_{k_1}+\dots+n_{k_A}=n_{m_1}+\dots+n_{m_A}$$

with $1 \leq k_i \leq N$, $1 \leq m_j \leq N$, repetitions allowed, which collapse down to 0 = 0.

The contribution to (*) stemming from those cases where all the k_i are distinct is simply

(**) $$A!\sum\left|c_{k_1}\right|^2\dots\left|c_{k_A}\right|^2,$$

where the summation is over all A-tuples (k_1, k_2, … k_A) with $1\leq k_i \leq N$ and $k_i \neq k_j$ if $i \neq j$.

Observe, however, that the total number of A-tuples (k_1, k_2, … k_A) with at least one pair of nondistinct k_i is $\leq B(A)\, N^{A-1}$. Therefore:

$$\sum\left|c_{k_1}\right|^2\dots\left|c_{k_A}\right|^2\leq M_0^{\;2A}B(A)N^{A-1},$$

where the summation is over all A-tuples (k_1, k_2, … k_A) with $1\leq k_i \leq N$ where $k_i = k_j$ for some $i \neq j$. By first forming the difference (*) – (**), we immediately see that

$$(*)=A!\sum\left|c_{k_1}\right|^2\dots\left|c_{k_A}\right|^2+O(1)N^{A-1}$$

with an implied constant depending solely on M_0 and A, and where the summation is over all A-tuples (k_1, k_2, … k_A) with $1 \leq k_i \leq N$. In other words:

$$(*)=A!\left(\sum_{j=1}^{N}|\,c_j\,|^2\right)^A+O(1)N^{A-1}.$$

By referring to (7), we finally discover that

(8) $$I_{AA}=A!\left(\sum_{j=1}^{N}|\,c_j\,|^2\right)^A+O(1)N^{A-1},$$

where the implied constant depends solely on M_0, A, and q. It follows at once that

(9) $$\lim_{N\to\infty} \frac{I_{AA}}{\sqrt{A_N}^{\,2A}} = 2^A A!$$

by virtue of (4).

This completes the proof of Theorem 1. ■

III. With the proof of theorem 1 laid out in detail as we have done, it is possible to derive theorem 1bis with not all that much more work.

One must finesse the statements of the lemmas to reflect a kind of multiplicative "weight" associated with $|c_k|$, so that *all* the earlier powers of N get systematically replaced by appropriate powers of

$$\sum_{k=1}^{N} |c_k|^2.^{*}$$

The key concept for the revamped lemmas is what we call "total c-mass." In lemma 2, for instance, this is *defined* to be

$$\sum |c_{k_1}|\cdots|c_{k_j}|$$

taken over all j-tuples $(k_1, k_2, \dots k_j)$, $1 \le k_i \le N$, pertinent for the original statement of lemma 2.

In lemma 3, the corresponding sum would be:

$$\sum |c_{k_1}|^{r_1}\cdots|c_{k_\mu}|^{r_\mu}.$$

We use a prime to indicate the *revamped form* of the respective lemmas. We also assume without loss of generality that

(10) $$|c_k| \le 1 \text{ for all j.}$$

Lemma 2' then asserts that:

$$[\text{total c-mass}] \le B(q, j)\left(\sum_{k=1}^{N} |c_k|^2\right)^{\frac{j-1}{2}}.$$

The proof follows the same general pattern as lemma 2 with a couple of key differences. Observe first that

(11) $$\max_{1\le k\le N} |c_k| \le \sqrt{\sum_{k=1}^{N} |c_k|^2}.$$

The cases j = 1 and j=2 are now readily seen to be ok. The *differences* appear in the induction step with $j \ge 3$. One assumes lemma 2' is proved for $1, \dots j-1$. To handle the case $n_{k_2}/n_{k_1} < (Mj - M)^{-1}$, one uses the fact that n_{k_1} must be unique and applies lemma 2' to $\pm n_{k_2} \pm \dots \pm n_{k_j} = n - n_{k_1}$.

The total c-mass here is thus, by (11), at most

$$B(q, j)\sqrt{\sum_{k=1}^{N} |c_k|^2}\cdot\sqrt{\sum_{k=1}^{N} |c_k|^2}^{\,j-2}.$$

The case $n_{k_2}/n_{k_1} \ge (Mj - M)^{-1}$ proceeds by writing $k_2 = k_1 - a$, and noting that $a \le B(q,j)$. One *subdivides* the subsequent estimation according to the value of a and the particular format of plus or minus signs. One applies lemma 2' to $\pm n_{k_3} \pm \dots \pm n_{k_j} = n - n_{k_1} \mp n_{k_2}$ for each choice of k_1. The total c-mass (in $\mathbb{Z}^j$) is thus seen to be at most

$$B(q, j)\left(\sum_{k_1=a+1}^{N} |c_{k_1}||c_{k_1-a}|\right)\sqrt{\sum_{k=1}^{N} |c_k|^2}^{\,j-3}.$$

By applying the Cauchy-Schwarz inequality to the k_1-sum, we see that the total c-mass is at most

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* The process we use will resemble threading a film into a camera, and then letting the film "ratchet" forward click-by-click. Matters will tend to take care of themselves once the receiving spindle has been primed; the phrase "divide and conquer" will also come to mind.

$$B(q,j)\sqrt{\sum_{k=1}^{N}|c_k|^2}^{\,j-1}$$

for each choice of a and each plus or minus format. Summing up all cases gives

$$B(q,j)\sqrt{\sum_{k=1}^{N}|c_k|^2}^{\,j-1}$$

which completes the induction.

Lemma 3' states that

$$[\text{total c-mass (in } \mathbb{Z}^j)] \le B(q,j)\left(1+\sqrt{\sum_{k=1}^{N}|c_k|^2}^{\,j-1}\right).$$

The proof is an easy mimic of that of lemma 3. The key observation is that, when $2 \le r_1 < j$,

$$\left(1+\sqrt{\sum_{k=1}^{N}|c_k|^2}^{\,j-r_1-1}\right)\left(\sum_{k_1=1}^{N}|c_{k_1}|^{r_1}\right) \le \sum_{k=1}^{N}|c_k|^2+\sqrt{\sum_{k=1}^{N}|c_k|^2}^{\,j-r_1+1} \le 2+2\sqrt{\sum_{k=1}^{N}|c_k|^2}^{\,j-1}$$

since $|c_{k_1}|^{r_1} \le |c_{k_1}|^2$ by (10).

Lemma 4' states that:

$$[\text{total c-mass (in } \mathbb{Z}^{A+B})] \le B(q,A,B)\left(1+\sqrt{\sum_{k=1}^{N}|c_k|^2}^{\,A+B-1}\right).$$

Similarly for lemmas 5' and 6' (with A = B). The proofs are again easy mimics.

One can now make an appropriate revamp of our earlier manipulations with I_{AB}. One uses lemmas 2' – 6' in place of lemmas 2 – 6.

One recovers (6) virtually immediately. For (7), one gets a bound of

$$B(q,A)\left(1+\left(\sum_{k=1}^{N}|c_k|^2\right)^{A-1/2}\right).$$

Note next that (*) on p.6 is a natural c-mass. The paragraph following (**) needs to be phrased in similar language. One finds that

$$\begin{bmatrix}\text{total c-mass for}\\ \text{k}_j \text{ not all distinct}\end{bmatrix} \le B(A)\left(\sum_{k=1}^{N}|c_k|^4\right)\left(\sum_{k=1}^{N}|c_k|^2\right)^{A-2} \le B(A)\left(\sum_{k=1}^{N}|c_k|^2\right)^{A-1}$$

since $|c_k|^4 \le |c_k|^2$ by (10). At once,

$$(*) = A!\left(\sum_{k=1}^{N}|c_k|^2\right)^{A}+O(1)\left(\sum_{k=1}^{N}|c_k|^2\right)^{A-1}.$$

Equation (8) on p.6 thus becomes

$$I_{AA} = A!\left(\sum_{k=1}^{N}|c_k|^2\right)^{A}+O(1)\left[1+\left(\sum_{k=1}^{N}|c_k|^2\right)^{A-1/2}\right],$$

with an implied constant dependent solely on A and q. (9) follows, and the proof of theorem 1bis is complete. ■

IV. In this section, we deal with the corollaries.

Corollary 2 follows from theorem 1-1bis by a standard result in probability theory. See, for instance, [B, pp. 406 – 408, 416 (30.6)]. The point is that a two-dimensional Gaussian is uniquely determined by its multi-moments. Our theorem thus assures us that

$$\lim_{N\to\infty} m\left\{x\in[0,1]:\frac{T_N(x)}{\sqrt{A_N}}\in E\right\}=\left(\frac{1}{\sqrt{2\pi}}\right)^2\iint_E e^{-\frac{1}{2}(u^2+v^2)}\,dudv$$

holds for every Jordan measurable set $E\subseteq\mathbb{R}^2$. Corollary 2 corresponds to $E=[\alpha,\beta]\times[\gamma,\delta]$.

Corollary 1 follows from theorem 1-1bis by recalling that

$$S_N(x)=\frac{T_N(x)+\overline{T_N(x)}}{2}$$

and using the binomial theorem.

V. Based on the results in [Z2, section 16.5], we suspect that, at least for *intervals* $J = [a, b]\subseteq[0, 1]$, it should be possible to prove that

$$\lim_{N\to\infty}\frac{1}{m(J)}\int_J\left(\frac{T_N(x)}{\sqrt{A_N}}\right)^A\left(\frac{\overline{T_N(x)}}{\sqrt{A_N}}\right)^B dx=\left(\frac{1}{\sqrt{2\pi}}\right)^2\iint_{\mathbb{R}^2}(u+iv)^A(u-iv)^B e^{-\frac{1}{2}(u^2+v^2)}\,dudv.$$

This would lead to J-analogs of corollaries 1 and 2. The J-analog of corollary 2 would correspond to [SZ, assertion (vi)]. See [Z1, section 5.6, lemma 6.5] for some hints about how one might proceed.

VI. Acknowledgements. I am extremely grateful to Professor Hejhal for valuable guidance. Section III is entirely due to him, and he helped streamline section II.

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