A MINIMAL CANTOR SET IN THE SPACE OF 3-GENERATED GROUPS

VOLODYMYR NEKRASHEVYCH

Abstract. We construct and study a family of 3-generated groups $\mathcal{D}_w$ parametrized by infinite binary sequences w. We show that two groups of the family are isomorphic if and only if the sequences are cofinal and that two groups can not be distinguished by finite sets of relations. We show a relation of the family with 2-dimensional holomorphic dynamics.

1. Introduction

1.1. **Space of finitely-generated groups.** Let $\mathfrak{G}_n$ be the set of all *marked n-generated groups*, i.e., the set of ordered tuples $(G, s_1, s_2, \ldots, s_n)$, where G is a group and $\{s_1, s_2, \ldots, s_n\} \subset G$ is its generating set. The set $\mathfrak{G}_n$ is a metrizable compact topological space with a natural topology (introduced in [7]). In this topology two marked groups are close if their (marked) Cayley graphs have large isomorphic balls.

One can also define the space $\mathfrak{G}_n$ in the following way. Let $F_n = \langle s_1, s_2, \ldots, s_n\rangle$ be a free non-abelian group of rank n. The set 2^{F_n} of subsets of F_n has a natural direct product (Tykhonov) topology. It is the topology defined by the basis of open *cylindrical sets* of the form

$$\mathcal{U}_{A,B} = \{N \subset F_n \ : \ A \subset N, B \cap N = \emptyset\},$$

where A and B are arbitrary finite subsets of F_n.

Then $\mathfrak{G}_n$ is homeomorphic to a closed subspace of 2^{F_n} where the homeomorphism maps a group G to the kernel of the natural epimorphism $F_n \to G$.

Some results on the structure of the space $\mathfrak{G}_n$ (and its natural direct limit $\mathfrak{G}_\infty$) are collected in [6]. In particular, one of the questions posed in [6] is construction of interesting closed subsets of $\mathfrak{G}_n$.

The space $\mathfrak{G}_n$ comes together with a natural Borel equivalence relation $\cong$ of group isomorphism. It is known (see [3]) that this relation is not *smooth*, i.e., does not have a Borel transversal. One of the aims of our paper is to show very explicit examples of *minimal Cantor sets* in $\mathfrak{G}_3$. We construct a closed $\cong$-invariant subset of $\mathfrak{G}_3$ homeomorphic to the Cantor set, such that the isomorphism classes are dense in it.

Namely, we construct a family $\{G_w = \langle \alpha_w, \beta_w, \gamma_w\rangle\}_{w \in \mathsf{X}^\omega}$, where

$$\mathsf{X}^\omega = \{x_1 x_2 \ldots \ : \ x_i \in \mathsf{X} = \{0, 1\}\}$$

is the space of infinite sequences over a binary alphabet (it is homeomorphic to the Cantor set) such that

(1) the map $w \mapsto (G_w, \alpha_w, \beta_w, \gamma_w)$ is a homeomorphism from X^ω to a subset of $\mathfrak{G}_3$,

(2) two groups G_{w_1} and G_{w_2} are isomorphic if and only if the sequences w_1 and w_2 are *cofinal*, i.e., are of the form $w_1 = v_1 w, w_2 = v_2 w$ for some infinite sequence $w \in \mathsf{X}^\omega$ and finite sequences $v_1, v_2 \in \mathsf{X}^*$ of equal length.

The groups G_w act on binary rooted trees. We denote by $\mathcal{D}_w$ the corresponding automorphism group of the rooted tree (i.e., the quotients of G_w by the kernels of the actions). The groups G_w and $\mathcal{D}_w$ coincide (i.e., G_w acts faithfully on the tree), when w has infinitely many zeros. If w has only a finite number of zeros, then G_w is an extension of an infinite direct cyclic groups of order 4 by the group $\mathcal{D}_{111\ldots}$.

The set $\{G_w\}_{w\in\mathsf{X}^\omega}$ is equal to the closure of the set

$$\{\mathcal{D}_w \ : \ w \text{ has infinitely many zeros}\}$$

in the space $\mathfrak{G}_3$.

Actually, the central topic of our paper is the family $\mathcal{D}_w$. We will also construct another family $\mathcal{R}_w$ of groups acting faithfully on the rooted tree (see the last section of the paper), which is closed in $\mathfrak{G}_3$ and there is no need of passing to the closure. But we do not know a complete answer to the isomorphism problem in the family $\{\mathcal{R}_w\}$ and many proofs are a bit longer for it (though follow the same pattern).

We have the following additional properties of the groups $\mathcal{D}_w$.

(1) The group $\mathcal{D}_{111\ldots}$ is isomorphic to one of the Grigorchuk groups G_ω.
(2) The group $\mathcal{D}_{000\ldots}$ is isomorphic to the iterated monodromy group of z^2+i.
(3) The groups $\mathcal{D}_w$ are branch and just-infinite.
(4) The closures of $\mathcal{D}_w$ in the automorphism group $\mathrm{Aut}(\mathsf{X}^*)$ of the rooted tree are conjugate.

1.2. **Self-similar families of groups.** The families $\{\mathcal{D}_w\}_{w\in\mathsf{X}^\omega}$ and $\{\mathcal{R}_w\}_{w\in\mathsf{X}^\omega}$ resemble the family $\{G_w\}$ of Grigorchuk groups, studied in [7]. It also becomes homeomorphic to a Cantor set after replacing a countable subset. However, the isomorphism classes of Grigorchuk groups G_w are finite (see Subsection 2.10.5 in [9]).

It is interesting that the intersection of the family $\{\mathcal{D}_w\}$ with the family of Grigorchuk groups is the set of the exceptional groups of $\{\mathcal{D}_w\}$ for which w has a finite number of zeros.

The following tentative definition generalizes some of the properties of these families.

For a definition of self-similar groups see [9] and the second section of our paper.

If G acts on a rooted tree T then the graph $G\backslash T$ of the orbits of the action is again a rooted tree. The preimages in T of the infinite rooted paths in $G\backslash T$ are precisely the minimal G-invariant rooted subtrees of T, which we call *G-irreducible components* of T. The boundaries of the G-irreducible components are the irreducible (i.e. minimal) components of the action of G on the boundary of the tree T.

Definition 1. A family of groups acting faithfully on rooted trees is said to be *self-similar* if there exists a self-similar group G such that the groups of the family coincide with the restrictions of the action of G onto the G-irreducible components of X^*. The group G is called *universal group* of the family.

The universal group will be our main tool of investigation of the families $\{\mathcal{D}_w\}$ and $\{\mathcal{R}_w\}$.

Many properties of the universal group remain to be mysterious even for the well studied family of Grigorchuk groups. One of the main questions is amenability of the universal groups of the Grigorchuk family and the families $\{\mathcal{D}_w\}$ and $\{\mathcal{R}_w\}$.

1.3. **Structure of the paper.** The second section contains all technical definitions and notions of the theory of self-similar groups, which will be used in the paper. The proofs and more details can be found in [9].

In the third section defines the family. We start from three automorphisms g_0, g_1, g_2 of the binary rooted tree $\mathsf{X}^* = \{0,1\}^*$. We consider next any triple of automorphisms h_0, h_1, h_2 such that h_i belongs to the conjugacy class of g_i in $\mathrm{Aut}(\mathsf{X}^*)$ (called *t-triples* in the paper). Examples of t-triples are $(\alpha_w, \beta_w, \gamma_w)$, which are inductively defined for every infinite sequence $w \in \mathsf{X}^\omega$ in Definition 8.

We prove then in Proposition 3.1 that any t-triple is simultaneously conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ with a unique triple $(\alpha_w, \beta_w, \gamma_w)$. We define $\mathcal{D}_w$ to be the group generated by α_w, β_w and γ_w.

A direct corollary of Proposition 3.1 is Proposition 3.2 claiming that every group $\mathcal{D}_w$ is conjugate to at most a countable set of groups of the family $\{\mathcal{D}_w\}$.

Section 4 studies the universal group $\mathcal{D}$ of the family, i.e., the group $F/\bigcap_{w\in\mathsf{X}^\omega}\mathcal{K}_w$, where $F = \langle \alpha, \beta, \gamma\rangle$ is the free group and $\mathcal{K}_w$ is the kernel of the natural epimorphism $\alpha \mapsto \alpha_w, \beta \mapsto \beta_w, \gamma \mapsto \gamma_w$ of F onto $\mathcal{D}_w$. In other words, $\mathcal{D}$ is the group generated by the diagonals $(\alpha_w)_{w\in\mathsf{X}^\omega}$, $(\beta_w)_{w\in\mathsf{X}^\omega}$ and $(\gamma_w)_{w\in\mathsf{X}^\omega}$ in the Cartesian product $\prod_{w\in\mathsf{X}^\omega}\mathcal{D}_w$.

The group $\mathcal{D}$ acts faithfully on the rooted tree $(\mathsf{X}\times\mathsf{X})^*$ and is the universal group of the family $\{\mathcal{D}_w\}$ in the sense of Definition 1. The action is defined by the recursions

$$\begin{aligned} \alpha &= \sigma \\ \beta &= (\alpha, \gamma, \alpha, \gamma) \\ \gamma &= (\beta, 1, 1, \beta)\,, \end{aligned}$$

where $\sigma = (12)(34)$.

We show that the group $\mathcal{D}$ is contracting (with a nucleus consisting of 35 elements) and find some relations in $\mathcal{D}$.

After that we embed $\mathcal{D}$ as a normal subgroup into a larger group $\widetilde{\mathcal{D}}$, whose action is level-transitive on $(\mathsf{X}\times\mathsf{X})^*$. The group $\widetilde{\mathcal{D}}$ is obtained from $\mathcal{D}$ by adjoining three new generators

$$\begin{aligned} a &= \pi, \\ b &= (a\alpha, a\alpha, c, c)\,, \\ c &= (b\beta, b\beta, b, b)\,, \end{aligned}$$

where $\pi = (13)(23)$.

If we identify the alphabet $\{1,2,3,4\}$ with $\mathsf{X}\times\mathsf{X}$ setting $1 \leftrightarrows (0,0), 2 \leftrightarrows (0,1), 3 \leftrightarrows (1,0), 4 \leftrightarrows (1,1)$, then the subgroup $\mathcal{D}$ of $\widetilde{\mathcal{D}}$ coincides with the group of elements acting trivially on the second coordinates of the letters (see Proposition 4.8). Hence, if we look only at the action of $\widetilde{\mathcal{D}}$ on the second coordinates of the letters, then we get an action of $\widetilde{\mathcal{D}}$ on the binary tree X^* with the kernel of the action equal to $\mathcal{D}$.

The quotient $H = \widetilde{\mathcal{D}}/\mathcal{D}$ is the subgroup of $\mathrm{Aut}(\mathsf{X}^*)$ generated by

$$a = \sigma, \quad b = (a, c), \quad c = (b, b).$$

It follows that if $w_1, w_2 \in \mathsf{X}^\omega$ belong to one H-orbit, then the groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ (see Proposition 4.6). This already implies that the sets of parameters $w \in \mathsf{X}^\omega$ corresponding to the conjugacy classes (and hence to the isomorphism classes) of $\mathcal{D}_w$ are dense in X^ω, since the action of H on X^ω is minimal.

We study then the group H and its action on X^ω in more detail. We prove that H is isomorphic to the group of all symmetries of the square lattice, that it is contracting, and that its orbits on X^ω coincide with the cofinality classes.

The last result of Section 4 is Proposition 4.12 implying that the group $\widetilde{\mathcal{D}}$ is contracting.

The isomorphism classes of the family $\{\mathcal{D}_w\}$ are completely described in Section 4.12. At first we prove in Theorem 5.1 that two groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate if and only if the sequences w_1 and w_2 are cofinal.

Then we construct (in Subsection 5.2) a collection of normal subgroups $\{\mathcal{E}_v\}_{v\in\mathsf{X}^*}$ of $\mathcal{D}$ (the kernels of the $|v|$th level wreath recursion for the groups $\mathcal{D}_{vw}$) and an action of H on $\mathcal{D}$ by automorphisms (coming from the action of $\widetilde{\mathcal{D}}$ on $\mathcal{D}$ by conjugation) such that

(1) $\mathcal{E}_{x_1x_2\ldots x_nx_{n+1}} > \mathcal{E}_{x_1x_2\ldots x_n}$;
(2) if $g(x_1x_2\ldots x_n) = y_1y_2\ldots y_n$, then $(\mathcal{E}_{x_1x_2\ldots x_n})^{g^{-1}} = \mathcal{E}_{y_1y_2\ldots y_n}$;
(3) if the sequence $x_1x_2\ldots \in \mathsf{X}^\omega$ has infinitely many zeros, then

$$\mathcal{D}/\bigcup_{n\geq 0}\mathcal{E}_{x_1x_2\ldots x_n}$$

is isomorphic to $\mathcal{D}_{x_1x_2\ldots}$.

We define $G_{x_1x_2\ldots} = \mathcal{D}/\bigcup_{n\geq 0}\mathcal{E}_{x_1x_2\ldots x_n}$ and prove in Theorem 5.10 that the map $w \mapsto G_w$ is a continuous map from X^ω to $\mathfrak{G}_3$.

Finally, we prove, using theory of branch groups, that two groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are isomorphic if and only if they are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$. This gives us a complete description of the isomorphism classes in the families $\{\mathcal{D}_w\}$ and $\{G_w\}$. Two groups in one family are isomorphic if and only if the corresponding sequences are cofinal (Theorem 6.1 and Proposition 6.2).

Section 7 gives a dynamical interpretation of the groups $\mathcal{D}_w$. We give only the main ideas and almost no proofs, since this will be a topic of a separate publication.

We show that the groups $\mathcal{D}_w$ can be interpreted as the iterated monodromy groups of non-autonomous (a.k.a. random) backward iterations of branched coverings of planes with the post-critical dynamics coinciding with the post-critical dynamics of $z^2 + i$.

Looking at the action of such branched coverings on the moduli space of the three-punctured plane, we get an interpretation of the index 2 subgroup $\langle ab, ac, \mathcal{D}\rangle$ of $\widetilde{\mathcal{D}}$ as the iterated monodromy group of the mapping

$$F : \begin{pmatrix} z \\ p \end{pmatrix} \mapsto \begin{pmatrix} \left(1 - \frac{2z}{p}\right)^2 \\ \left(1 - \frac{2}{p}\right)^2 \end{pmatrix}$$

on $\mathbb{CP}^2$.

Using this interpretation one can show that the connected components of the limit space of $\mathcal{D}$ are homeomorphic to the Julia sets of the forward non-autonomous iterations of quadratic polynomials (which are the first coordinates of the iteration of F as functions of z). We give some examples of these Julia sets on Figure 6.

Last section of the paper describes a similar family of groups $\{\mathcal{R}_w\}$, which can be also constructed from non-autonomous iterations, but starting from another post-critical dynamics.

2. Preliminaries

2.1. **Rooted trees, automata and self-similar groups.** Let X be a finite set (called the *alphabet*) and denote by X^* the free monoid generated by X, i.e., the set of finite words over the alphabet X.

The set X^* is considered to be a rooted tree with the root the empty word $\varnothing$ and two words connected by an edge if and only if they are of the form v, vx for $v \in \mathsf{X}^*$ and $x \in \mathsf{X}$.

The set X^n of words of length n is called the nth level of the rooted tree X^*.

We denote by $\mathrm{Aut}(\mathsf{X}^*)$ the group of all automorphisms of the rooted tree X^*. If G is a group acting by automorphisms on X^*, then the *nth level stabilizer* is the subgroup of elements of G acting trivially on the nth level X^n of X^*.

The sequence of the nth level stabilizers of $\mathrm{Aut}(\mathsf{X}^*)$ is a fundamental system of neighborhoods of identity defining a natural profinite topology on $\mathrm{Aut}(\mathsf{X}^*)$, coinciding with the topology of pointwise convergency on X^*.

A bijection $g : \mathsf{X}^* \to \mathsf{X}^*$ is an automorphism if and only if it preserves the levels and for all $v, u \in \mathsf{X}^*$ there exists $w \in \mathsf{X}^*$ such that $g(vu) = g(v)w$. It is easy to see then that then for every $v \in \mathsf{X}^*$ the map $u \mapsto w$ is also an automorphism of X^*. We will denote it $g|_v$ and call *restriction of g in v*.

Definition 2. A subgroup $G \le \mathrm{Aut}(\mathsf{X}^*)$ is *self-similar* (or *state-closed*) if for every $g \in G$ and $x \in \mathsf{X}$ the restriction $g|_x$ also belongs to G.

Every automorphism $g \in \mathrm{Aut}(\mathsf{X}^*)$ is uniquely determined by the permutation $\alpha \in \mathfrak{S}(\mathsf{X})$ it performs on the first level of the tree and the function $x \mapsto g|_x$. The described bijection between $\mathrm{Aut}(\mathsf{X}^*)$ and the set $\mathrm{Aut}(\mathsf{X}^*)^{\mathsf{X}} \times \mathfrak{S}(\mathsf{X})$ is an isomorphism between $\mathrm{Aut}(\mathsf{X}^*)$ and the permutational wreath product $\mathfrak{S}(\mathsf{X}) \wr \mathrm{Aut}(\mathsf{X}^*)$. We identify $\mathrm{Aut}(\mathsf{X}^*)$ with $\mathfrak{S}(\mathsf{X}) \wr \mathrm{Aut}(\mathsf{X}^*)$ by this bijection and write

$$g = \pi(g_1, g_2, \dots, g_d),$$

where $\pi \in \mathfrak{S}(\mathsf{X})$ is the action of g on the first level and $g_i = g|_{x_i}$ for a fixed enumeration $\{x_1, x_2, \dots, x_d\} = \mathsf{X}$ of the alphabet.

In this identification the group $\mathfrak{S}(\mathsf{X}) < \mathfrak{S}(\mathsf{X}) \wr \mathrm{Aut}(\mathsf{X}^*)$ acts on the tree by the *rigid action*

$$\pi(y_1 y_2 \dots y_n) = \pi(y_1) y_2 \dots y_n,$$

and the base $\mathrm{Aut}(\mathsf{X}^*)^{\mathsf{X}} < \mathfrak{S}(\mathsf{X}) \wr \mathrm{Aut}(\mathsf{X}^*)$ is identified with the pointwise stabilizer of the first level of X^* and acts on the tree by the rule

$$(g_1, \dots, g_d)(x_{i_1} x_{i_2} \dots x_{i_n}) = x_{i_1} g_{i_1}(x_{i_2} \dots x_{i_n}).$$

The multiplication rule in $\mathfrak{S}(\mathsf{X}) \wr \mathrm{Aut}(\mathsf{X}^*)$ is

$$\pi_1(g_1, \dots, g_d) \cdot \pi_2(h_1, \dots, h_d) = \pi_1 \pi_2 (g_{\pi_2(1)} h_1, \dots, g_{\pi_2(d)} h_d),$$

where $\pi_2(i)$ is such that $\pi_2(x_i) = x_{\pi_2(i)}$. Note that all groups in our paper act from the left on words.

Definition 3. The isomorphism $\mathrm{Aut}(\mathsf{X}^*) \to \mathfrak{S}(\mathsf{X}) \wr \mathrm{Aut}(\mathsf{X}^*)$ is called the *wreath recursion*.

If $G = \langle g_1, \dots, g_m \rangle$ is a finitely generated self-similar group, then its action on X^* is uniquely determined by the images of its generators under the wreath recursion, i.e., by the relations

$$\begin{array}{rcl} g_1 & = & \pi_1(g_{11}, \dots, g_{1d}), \\ g_2 & = & \pi_2(g_{21}, \dots, g_{2d}), \\ \vdots & & \\ g_m & = & \pi_m(g_{m1}, \dots, g_{md}), \end{array}$$

where g_{ij} are group words in $g_1, g_2, \dots, g_m$.

If we have $g_{ij} \in \{g_1, \dots, g_m\}$ then the set $\{g_1, \dots, g_m\}$ can be considered as a set of states of an *automaton*, which being in a state g_i and reading on the input a letter x_j gives on the output the letter $\pi_i(x_j)$ and going to the state $g_{ij} = g_i|_{x_j}$.

We draw such automata as graphs with the vertices identified with its states and an arrow from g to h labeled by a pair $(x, y) \in \mathsf{X} \times \mathsf{X}$ if $g(x) = y$ and $g|_x = h$.

2.2. **Wreath recursion and permutational bimodules.** A self-similar group $G \le \mathrm{Aut}(\mathsf{X}^*)$ is uniquely determined by the associated *wreath recursion*

$$\psi : G \to \mathfrak{S}(\mathsf{X}) \wr G,$$

which is the restriction of the isomorphism $\mathrm{Aut}(\mathsf{X}^*) \to \mathfrak{S}(\mathsf{X}) \wr \mathrm{Aut}(\mathsf{X}^*)$.

We can interpret a homomorphism $\psi : G \to \mathfrak{S}(\mathsf{X}) \wr G$ as a structure of a *permutational bimodule* on the set $\mathsf{X} \times G$. Namely, we consider the set $\mathsf{X} \times G$ to be a free $|\mathsf{X}|$-dimensional right G-set with the action

$$(x \cdot h) \cdot g = x \cdot (gh).$$

Then the group $\mathfrak{S}(\mathsf{X}) \wr G$ coincides with the group of automorphisms of the left G-set $\mathsf{X} \times G$. Its elements $\pi(g_x)_{x\in\mathsf{X}}$ act on $\mathsf{X} \times G$ by

$$\pi(g_x)_{x\in\mathsf{X}} \cdot (x \cdot g) = \pi(x) \cdot g_x g.$$

The homomorphism ψ defines then a left G-action on $\mathsf{X} \times G$, which commutes with the right one. This action is given by

$$g \cdot (x \cdot h) = y \cdot g|_x h,$$

where $y = \pi(g)$ for $\psi(g) = \pi\,(g|_x)_{x\in\mathsf{X}}$.

If ψ is the wreath recursion of a self-similar group, then the bimodule $\Phi = \mathsf{X} \times G$ is called *self-similarity bimodule*. We can identify an element $x \cdot g$ of the self-similarity bimodule with the map $\mathsf{X}^* \to \mathsf{X}^*$ given by

$$(x \cdot g)(v) = xg(v).$$

Then the left and the right actions of G on Φ become usual compositions of transformations.

The wreath recursion (and hence the self-similar action) can be reconstructed from the self-similarity bimodule in the following way.

Definition 4. Let Φ be a permutational G-bimodule, i.e., a set with commuting left and right actions of G. Suppose that the right action is free, i.e., equality $m \cdot g = m$ implies $g = 1$. Then a *basis* of Φ is any transversal of the right G-orbits on Φ.

For example, the set $\{x \cdot 1 \ : \ x \in \mathsf{X}\}$ is a basis of the self-similarity bimodule $\mathsf{X} \times G$.

If X is a basis of Φ, then every element $m \in \Phi$ can be uniquely written in the form $x \cdot g$ for $x \in \mathsf{X}$ and $g \in G$. In particular, for every $x \in \mathsf{X}$ and $g \in G$ there exist $y \in \mathsf{X}$ and $h \in G$ such that

$$g \cdot x = y \cdot h.$$

The map $x \mapsto y$ is a permutation and we denote $y = g(x)$. The element h is called *restriction of* g *in* x and is denoted $g|_x$.

Then the map $g \mapsto \pi(g|_x)_{x \in \mathsf{X}}$ is a homomorphism from G to $\mathfrak{S}(\mathsf{X}) \wr G$. It coincides with the natural homomorphism mapping g to the automorphism $m \mapsto g \cdot m$ of the right G-space Φ. This homomorphism is called the *wreath recursion* associated with the bimodule Φ (and the basis X).

Hence every basis X of a bimodule Φ defines a wreath recursion $G \to \mathfrak{S}(\mathsf{X}) \wr G$ and the associated self-similar action of G on X^*.

If we change the basis, then we will change the wreath recursion, but the self-similar action will be conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ to the original one (see Proposition 2.3.4 in [9]).

We need also the following operations over G-bimodules.

Definition 5. Let Φ_1 and Φ_2 be permutational G-bimodules. Their *tensor product* $\Phi_1 \otimes \Phi_2$ is the quotient of the direct product $\Phi_1 \times \Phi_2$ by the equivalence relation

$$(m_1 \cdot g) \times m_2 = m_1 \cdot (g \cdot m_2).$$

The element $m_1 \times m_2$ of $\Phi_1 \otimes \Phi_2$ is written $m_1 \otimes m_2$. The action of G on $\Phi_1 \otimes \Phi_2$ is given by

$$g \cdot (m_1 \otimes m_2) = (g \cdot m_1) \otimes m_2, \quad (m_1 \otimes m_2) \cdot g = m_1 \otimes (m_2 \cdot g).$$

The *direct sum* $\Phi_1 \oplus \Phi_2$ is the disjoint union of the sets Φ_1, Φ_2 with the original actions of G on each of the sets Φ_i.

If X_i are bases of Φ_i for $i = 1, 2$, then $\mathsf{X}_1 \otimes \mathsf{X}_2$ is a basis of $\Phi_1 \otimes \Phi_2$.

In particular, for every n the set $\mathsf{X}^{\otimes n}$ is a basis of the tensor power $\Phi^{\otimes n}$ of the self-similarity bimodule $\Phi = \mathsf{X} \times G$. We have an equality

$$g \cdot x_1 \otimes x_2 \otimes \cdots \otimes x_n = y_1 \otimes y_2 \otimes \cdots \otimes y_n \cdot h$$

if and only if

$$g(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n, \text{ and } g|_{x_1 x_2 \ldots x_n} = h.$$

We will use a common notation for permutational bimodules and write

$$g \cdot x_1 x_2 \ldots x_n = y_1 y_2 \ldots y_n \cdot h,$$

$$g(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n, \quad g|_{x_1 x_2 \ldots x_n} = h,$$

if, in the case of a self-similar action, $g(x_1 x_2 \ldots x_n w) = y_1 y_2 \ldots y_n h(w)$ for all $w \in \mathsf{X}^*$, and, in the case of permutational bimodules, if $g \cdot x_1 \otimes x_2 \otimes \cdots \otimes x_n = y_1 \otimes y_2 \otimes \cdots \otimes y_n \cdot h$.

2.3. **Contracting actions and limit spaces.**

Definition 6. A self-similar action of a group G on a tree X^* (a permutational G bimodule Φ with free right action and a fixed basis X) is said to be *contracting* if there exists a finite subset $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for all words v of length more than n.

The *nucleus* of a contracting action (of a contracting bimodule) is the minimal set $\mathcal{N}$ satisfying the conditions of the previous definition. It is not hard to see that the nucleus is unique.

If G is generated by a finite set $S = S^{-1}$, then a finite set $A \subset G$ contains the nucleus if and only if for every $g \in A$ and s in $S \cup \{1\}$ there exists n such that $gs|_v \in A$ for all words of length more than n.

If a bimodule is contracting with respect to some basis, then it is contracting with respect to any basis. However, the nucleus depends on the basis.

If we have a contracting group G, then its *limit space* $\mathcal{J}_G$ is defined as the quotient of the topological space $\mathsf{X}^{-\omega}$ of the left-infinite sequences $\ldots x_2 x_1$ with respect to the *asymptotic equivalence relation*. Two sequences $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ are asymptotically equivalent if there exists a sequence $g_n \in G$ assuming a finite set of values such that $g_n(x_n x_{n-1} \ldots x_1) = y_n y_{n-1} \ldots y_1$ for all n.

The limit space of a contracting self-similar group is a metrizable compact space of finite topological dimension.

More on contracting groups and their limit spaces see [9].

Notations. In most cases X is the binary alphabet $\{0, 1\}$, $\mathfrak{S}(\mathsf{X}) = \{1, \sigma = (0, 1)\}$ is symmetric group. The actions of groups on words, infinite sequences and trees is left and the image of v under action of $g \in G$ is denoted $g(v)$. The action of group automorphisms ϕ is right and is denoted g^ϕ. We also write $g^h = h^{-1} g h$, and $[g, h] = g^{-1} h^{-1} g h$.

3. Definition of the family

Consider three automorphisms g_0, g_1, g_2 of the rooted binary tree, which are defined by the recursions

$$g_0 = \sigma, \quad g_1 = (g_0, g_2), \quad g_2 = (1, g_1),$$

where $\sigma = (0, 1)$ is the transposition. In other terms, these automorphisms are the states of the four-state automaton shown on Figure 1.

The action of the automorphisms g_0, g_1, g_2 are shown schematically on Figure 2.

Note that the group generated by these automorphisms is one of the Grigorchuk groups G_ω, see [7]. This group (in particular its growth) was later studied by A. Erschler in [4].

Two automorphisms of a rooted tree are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ if and only if their *tree of orbits* are isomorphic (see [5]). Here the tree of orbits of an automorphism g is the tree of orbits of the cyclic group generated by g, where each vertex is labelled by the cardinality of the respective orbit. Figure 3 shows the trees of orbits of the automorphisms g_0, g_1, g_2 (without vertex labels).

Definition 7. A *t-triple* is a triple (h_0, h_1, h_2) of automorphisms of the tree X^*, conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ (not necessary by the same conjugators) with the automorphisms g_0, g_1, g_2, respectively.

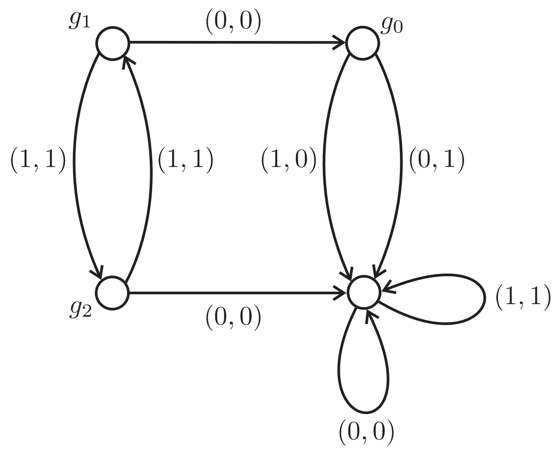

FIGURE 1. Automaton

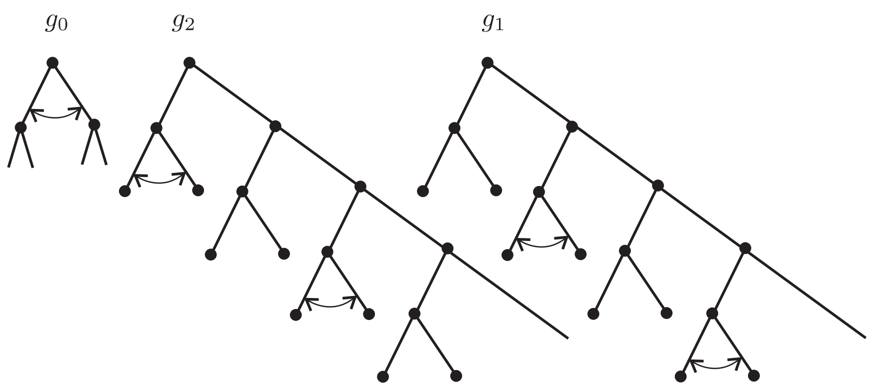

FIGURE 2. Portraits of g_0, g_1 and g_2

Definition 8. Let $w \in \mathsf{X}^\omega$ be an infinite binary sequence. Define the following elements $\alpha_w, \beta_w, \gamma_w$ of $\mathrm{Aut}(\mathsf{X}^*)$ recurrently

$$\begin{aligned}\alpha_w &= \sigma,\\ \beta_w &= \big(\alpha_{s(w)}, \gamma_{s(w)}\big)\\ \gamma_w &= \begin{cases}\big(\beta_{s(w)}, 1\big) & \text{if the first letter of } w \text{ is } 0,\\ \big(1, \beta_{s(w)}\big) & \text{if the first letter of } w \text{ is } 1.\end{cases}\end{aligned}$$

Here $s(w)$ is the shift of the sequence w, i.e., the sequence obtained by deletion of its first letter.

For example, the elements g_0, g_1, g_2 are equal to $\alpha_w, \beta_w, \gamma_w$, respectively, when $w = 111\ldots$.

It is easy to see that $(\alpha_w, \beta_w, \gamma_w)$ is an t-triple.

Proposition 3.1. *Let (h_0, h_1, h_2) be an arbitrary t-triple. Then there exists a unique sequence $W(h_0, h_1, h_2) = w \in \mathsf{X}^\omega$ such that the triple (h_0, h_1, h_2) is simultaneously conjugate with the triple $(\alpha_w, \beta_w, \gamma_w)$, i.e., $\alpha_w = h_0^h$, $\beta_w = h_1^h$ and $\gamma_w = h_2^h$ for some $h \in \mathrm{Aut}(\mathsf{X}^*)$.*

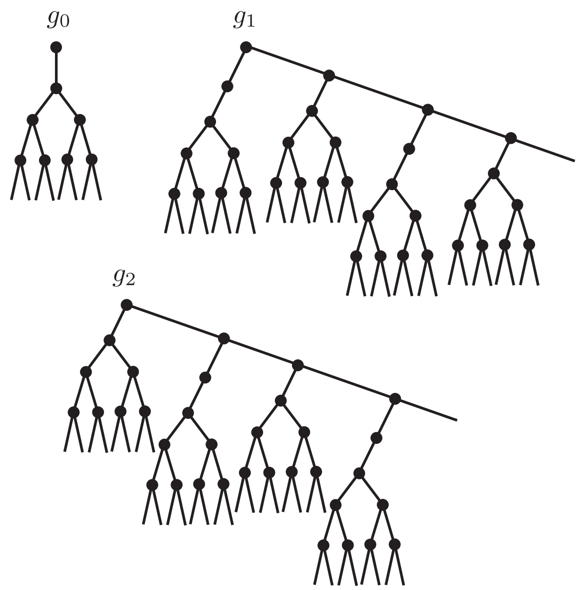

FIGURE 3. Orbit trees of g_0, g_1 and g_2

Proof. We will use the following notation, introduced by S. Sidki. For $g \in \mathrm{Aut}(\mathsf{X}^*)$ denote by $g^{(1)}$ the automorphism (g, g). Inductively, $g^{(k)}$ is defined then by the condition $g^{(k+1)} = \left(g^{(k)}\right)^{(1)}$. Then $g^{(k)}$ belongs to the kth level stabilizer and therefore the product $a_0 a_1^{(1)} a_2^{(2)} \cdots$ converges in $\mathrm{Aut}(\mathsf{X}^*)$ for any sequence $a_0, a_1, \ldots$.

If (h_0, h_1, h_2) is a t-triple, then h_0 is of the form $\sigma(g, g^{-1})$, since it is conjugate to σ. The element h_1 is either of the form (h_0', h_2') or of the form (h_2', h_0'), where h_0' and h_2' are conjugate to g_0 and g_2, respectively. Conjugating the triple by σ, if necessary, we may assume that $h_1 = (h_0', h_2')$. Then the element h_2 is either of the form $(h_2', 1)$, or of the form $(1, h_2')$. The only possibility for the first letter of $W(h_0, h_1, h_2)$ is 0 in the first case and 1 in the second. We see that if the sequence $W(h_0, h_1, h_2)$ exists, then its first letter is uniquely defined.

If the first letter of $W(h_0, h_1, h_2)$ is 0, then we conjugate the triple by the automorphism $a_0 = (1, g)$ and get

$$h_0^{a_0} = \sigma, \quad h_1^{a_0} = (h_0', {h_2'}^{g}), \quad h_2^{a_0} = (h_1', 1). \tag{1}$$

If the first letter of $W(h_0, h_1, h_2)$ is 1, then we may conjugate by $a_0 = (g^{-1}, 1)$ and get

$$h_0^{a_0} = \sigma, \quad h_1^{a_0} = ({h_0'}^{g^{-1}}, h_2'), \quad h_2^{a_0} = (1, h_1'). \tag{2}$$

We see that in any case there exists a_0 such that

$$h_0^{a_0} = \sigma, \quad h_1^{a_0} = (\widetilde{h}_0, \widetilde{h}_1), \quad h_2^{a_0} = (\widetilde{h}_1, 1), \text{ or } (1, \widetilde{h}_1),$$

where $(\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3)$ is a t-triple.

We repeat now the same procedure for $\widetilde{h}_0, \widetilde{h}_1$ and $\widetilde{h}_2$ and find the second possible letter of the sequence $W(h_0, h_1, h_2)$ and a conjugator a_1, transforming the automorphisms $\widetilde{h}_0, \widetilde{h}_1$ and $\widetilde{h}_2$ into the nice form. Note that then

$$h_0^{a_0 a_1^{(1)}} = \sigma, \quad h_1^{a_0 a_1^{(1)}} = (\widetilde{h}_0^{a_1}, \widetilde{h}_1^{a_1}), \quad h_2^{a_0 a_1^{(1)}} = (\widetilde{h}_1^{a_1}, 1), \text{ or } (1, \widetilde{h}_1^{a_1}).$$

We continue the procedure further (now with the states of $\widetilde{h}_i^{a_1}$), and at the end we will get all letters of the word $w = W(h_0, h_1, h_2)$ and a sequence $a_0, a_1, a_2, \ldots \in \mathrm{Aut}(\mathsf{X}^*)$ such that

$$h_0^{a_0 a_1^{(1)} a_2^{(2)} a_3^{(3)} \cdots} = \alpha_w, \quad h_1^{a_0 a_1^{(1)} a_2^{(2)} a_3^{(3)} \cdots} = \beta_w, \quad h_2^{a_0 a_1^{(1)} a_2^{(2)} a_3^{(3)} \cdots} = \gamma_w.$$

□

Definition 9. Let us denote by $\mathcal{D}_w$ the group, generated by α_w, β_w and γ_w.

Proposition 3.2. *For every word $w \in \mathsf{X}^\omega$ there exists at most a countable set of words $w' \in \mathsf{X}^\omega$ such that $\mathcal{D}_w$ is conjugate to $\mathcal{D}_{w'}$ in* $\mathrm{Aut}(\mathsf{X}^*)$.

Proof. If $\mathcal{D}_w$ and $\mathcal{D}_{w'}$ are conjugate, then there exists a generating set h_0, h_1, h_2 of $\mathcal{D}_w$ such that the triple (h_0, h_1, h_2) is conjugate to the triple $(\alpha_{w'}, \beta_{w'}, \gamma_{w'})$. The word w' is determined by the triple h_0, h_1, h_2 uniquely, by Proposition 3.1. But the number of possible generating sets $\{h_0, h_1, h_2\}$ of $\mathcal{D}_w$ is at most countable. □

4. Universal group of the family

4.1. **The construction.** Consider the following group $\mathcal{D} = \langle \alpha, \beta, \gamma \rangle$ acting on the 4-regular tree $\mathsf{T} = \{1, 2, 3, 4\}^*$:

$$\begin{aligned} \alpha &= \sigma \\ \beta &= (\alpha, \gamma, \alpha, \gamma) \\ \gamma &= (\beta, 1, 1, \beta), \end{aligned}$$

where $\sigma = (12)(34)$.

Let us identify the letters $1, 2, 3$ and 4 with the pairs $(0,0), (1,0), (0,1)$ and $(1,1)$, respectively. Then σ is the permutation $(0, y) \leftrightarrow (1, y)$. Hence, the elements of the group $\mathcal{D}$ leave in every word

$$(x_1, y_1)(x_2, y_2) \ldots (x_n, y_n)$$

the second coordinates $y_1, y_2, \ldots, y_n$ unchanged.

Let us fix some infinite word $w = y_1 y_2 \ldots \in \mathsf{X}^\omega$. Then the set T_w of all vertices of the rooted tree $(\mathsf{X} \times \mathsf{X})^* = \mathsf{T}$ of the form

$$(x_1, y_1)(x_2, y_2) \ldots (x_n, y_n)$$

is a sub-tree isomorphic to the binary tree X^*. The isomorphism is the map

$$L_w : (x_1, y_1)(x_2, y_2) \ldots (x_n, y_n) \mapsto x_1 x_2 \ldots x_n.$$

The tree T_w is $\mathcal{D}$-invariant. It follows directly from the recursions, that when we conjugate the action of α, β, γ on T_w by the isomorphism L_w, then we get the automorphisms $\alpha_w, \beta_w, \gamma_w$ of X^*.

We get thus the following description of $\mathcal{D}$.

Proposition 4.1. *The restriction of the action of $\mathcal{D}$ onto the subtree T_w is conjugate with the action of $\mathcal{D}_w$ on the binary tree.*

The group $\mathcal{D}$ is isomorphic to the quotient $F \big/ \bigcap_{w \in \mathsf{X}^\omega} \mathcal{K}_w$, where F is the free group generated by letters α, β, γ and $\mathcal{K}_w$ is the kernel of the canonical epimorphism $F \to \mathcal{D}_w$ mapping α, β and γ to α_w, β_w and γ_w, respectively.

In other words, $\mathcal{D}$ is isomorphic to the group, generated by the diagonal elements

$$\alpha = (\alpha_w)_{w\in\mathsf{X}^\omega}, \beta = (\beta_w)_{w\in\mathsf{X}^\omega}, \gamma = (\gamma_w)_{w\in\mathsf{X}^\omega}$$

of the Cartesian product $\prod_{w\in\mathsf{X}^\omega} \mathcal{D}_w$.

It is easy to prove by induction that every group $\mathcal{D}_w$ is level-transitive. Consequently, the quotient of the tree $(\mathsf{X}\times\mathsf{X})^*$ by the action of the group $\mathcal{D}$ is binary and is naturally identified with X^*. Two words $(x_1, y_1)\ldots(x_n, y_n)$ and $(x_1', y_1')\ldots(x_n', y_n')$ belong to one $\mathcal{D}$-orbit if and only if $y_i = y_i'$. If w is an infinite path in the quotient-tree, then its preimage in $(\mathsf{X}\times\mathsf{X})^*$ is the binary tree T_w.

Let us introduce the following notation. If g is an element of $\mathcal{D}$, then we denote by g_w its image in $\mathcal{D}_w$, i.e, the element of $\mathcal{D}_w$ obtained by restricting the action of g onto T_w and then conjugating by the natural isomorphism $T_w \to \mathsf{X}^*$.

The homomorphism $g\mapsto g_w$ is the extension of the map $\alpha\mapsto\alpha_w$, $\beta\mapsto\beta_w$ and $\gamma\mapsto\gamma_w$.

4.2. **Nucleus of $\mathcal{D}$.** It is easy to see that $\alpha,\beta,\gamma\in\mathcal{D}$ are elements of order 2.

Consequently, $\alpha\gamma$ is of order 4, since

$$(\alpha\gamma)^2 = \sigma(\beta,1,1,\beta)\sigma(\beta,1,1,\beta) = (\beta,\beta,\beta,\beta).$$

We get

$$(\alpha\beta)^2 = \sigma(\alpha,\gamma,\alpha,\gamma)\sigma(\alpha,\gamma,\alpha,\gamma) = (\gamma\alpha,\alpha\gamma,\gamma\alpha,\alpha\gamma),$$

hence $\alpha\beta$ is of order 8.

Then $\beta\gamma = (\alpha\beta,\gamma,\alpha,\gamma\beta)$, which implies that $\beta\gamma$ also is of order 8. Consequently, $\langle\beta,\gamma\rangle$ is isomorphic to the dihedral group D_8 of order 16.

We get thus the following subgroups of $\mathcal{D}$:

$$\langle\alpha,\beta\rangle\cong D_8,\quad \langle\alpha,\gamma\rangle\cong D_4,\quad \langle\beta,\gamma\rangle\cong D_8.$$

Proposition 4.2. *The group $\mathcal{D}$ is contracting with the nucleus*

$$\mathcal{N} = \langle\beta,\gamma\rangle\cup\langle\gamma,\alpha\rangle\cup\langle\alpha,\beta\rangle.$$

The nucleus contains 35 elements.

Proof. We have the following decompositions of the elements of the set $\mathcal{N}$.

If g is an element of $\langle\beta,\gamma\rangle$, then its image under the wreath recursion belongs to the direct product $\langle\alpha,\beta\rangle\times\langle\gamma\rangle\times\langle\alpha\rangle\times\langle\beta,\gamma\rangle$. Where the projection on the first coordinate is the isomorphism $\beta\mapsto\alpha,\gamma\mapsto\beta$, the projection on the second coordinate is the homomorphism $\beta\mapsto\gamma,\gamma\mapsto 1$, the projection on the third coordinate is the homomorphism $\beta\mapsto\alpha,\gamma\mapsto 1$ and the projection on the fourth coordinate is the isomorphism $\beta\mapsto\gamma,\gamma\mapsto\beta$.

The wreath decomposition of $\langle\alpha,\gamma\rangle$ is embedding $D_4\hookrightarrow C_2\wr C_2$ given by

$$\alpha\mapsto\sigma = (12)(34),\quad \gamma\mapsto(\beta,1,1,\beta).$$

The wreath decomposition of $\langle\alpha,\beta\rangle$ is embedding $D_8\hookrightarrow C_2\wr D_4$ given by

$$\alpha\mapsto\sigma,\quad \beta\mapsto(\alpha,\gamma,\alpha,\gamma).$$

We see that the set $\mathcal{N}$ is state-closed.

It is sufficient now to prove that for any $g\in\mathcal{N}$ there exists n such that the restrictions $(g\beta)|_v, (g\gamma)|_v$ belong to $\mathcal{N}$ for all $v\in\mathsf{X}^n$. Moreover, it is sufficient to consider the case $g\in\langle\alpha,\gamma\rangle$ when considering the restriction $(g\beta)|_v$ and $g\in\langle\alpha,\beta\rangle$ for the restriction $(g\gamma)|_v$.

But the first level decomposition of $g\beta$ for $g \in \langle \alpha, \gamma \rangle$ belongs to

$$(\langle \alpha, \beta \rangle \times \langle \gamma, \beta \rangle \times \langle \alpha, \beta \rangle \times \langle \gamma, \beta \rangle) \cdot \langle \sigma \rangle,$$

hence we may take $n = 1$.

The first level restrictions of the elements $g\gamma$ for $g \in \langle \alpha, \beta \rangle$ are of the form $h\beta$ or h, where $h \in \langle \alpha, \gamma \rangle$. Hence, by the above, we can take $n = 2$ in this case. □

Let us see which of the relations between α, β, γ hold in separate groups $\mathcal{D}_w$.

Proposition 4.3. *For every $w \in \mathsf{X}^\omega$ the elements $\alpha_w, \beta_w, \gamma_w$ are of order 2, $\alpha_w\gamma_w$ is of order 4 and $\alpha_w\beta_w$ is of order 8. If $w = 111\ldots$, then $\beta_w\gamma_w$ is of order 2, otherwise it is of order 8.*

Proof. The statements about the order of $\alpha_w, \beta_w, \gamma_w, \alpha_w\gamma_w$ and $\alpha_w\beta_w$ are easy to check.

Let us consider the element $\beta_w\gamma_w$. If the first letter of w is 0, then $\beta_w\gamma_w = (\beta_{s(w)}\alpha_{s(w)}, \gamma_{s(w)})$ is of order 8. If the first letter of w is 1, then $\beta_w\gamma_w = (\alpha_{s(w)}, \gamma_{s(w)}\beta_{s(w)})$ is of order equal to the order of $\gamma_{s(w)}\beta_{s(w)}$. Consequently, if w has at least one zero, then $\beta_w\gamma_w$ is of order 8. On the other hand, if $w = 111\ldots$, then

$$\begin{aligned}
\beta_{111\ldots}\gamma_{111\ldots} &= (\alpha_{111\ldots}, \gamma_{111\ldots}\beta_{111\ldots}),\\
\gamma_{111\ldots}\beta_{111\ldots} &= (\alpha_{111\ldots}, \beta_{111\ldots}\gamma_{111\ldots}),
\end{aligned}$$

which implies that $\beta_{111\ldots}$ and $\gamma_{111\ldots}$ commute and their product is of order 2. □

4.3. **Overgroup $\widetilde{\mathcal{D}}$ and conjugacy of the groups $\mathcal{D}_w$.** Consider now the group $\widetilde{\mathcal{D}}$ generated by the automorphisms

$$\begin{aligned}
\alpha &= \sigma, & a &= \pi,\\
\beta &= (\alpha, \gamma, \alpha, \gamma), & b &= (a\alpha, a\alpha, c, c),\\
\gamma &= (\beta, 1, 1, \beta), & c &= (b\beta, b\beta, b, b),
\end{aligned}$$

where $\sigma = (12)(34)$ is the permutation $(0, x) \leftrightarrow (1, x)$ and $\pi = (13)(23)$ is the permutation $(x, 0) \leftrightarrow (x, 1)$.

Proposition 4.4. *The following relations hold.*

$$\begin{aligned}
\alpha^a &= \alpha, & \alpha^b &= \alpha, & \alpha^c &= \alpha,\\
\beta^a &= \beta, & \beta^b &= \beta, & \beta^c &= \beta^\gamma,\\
\gamma^a &= \gamma^\alpha, & \gamma^b &= \gamma^\beta, & \gamma^c &= \gamma.
\end{aligned}$$

Proof. The following is obtained directly from the recursions.

$$\begin{aligned}
\alpha^a &= \alpha, & \alpha^b &= \alpha, & \alpha^c &= \alpha\\
\beta^a &= \beta, & \beta^b &= (\alpha^{a\alpha}, \gamma^{a\alpha}, \alpha^c, \gamma^c), & \beta^c &= (\alpha^{b\beta}, \gamma^{b\beta}, \alpha^b, \gamma^b)\\
\gamma^a &= (1, \beta, \beta, 1) = \gamma^\alpha, & \gamma^b &= (\beta^{a\alpha}, 1, 1, \beta^c), & \gamma^c &= (\beta^{b\beta}, 1, 1, \beta^b).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\beta^b &= (\alpha, \gamma, \alpha, \gamma^c),\\
\beta^{b\beta} &= (\alpha, \gamma, \alpha, \gamma^{c\gamma}),\\
\gamma^{c\gamma} &= (\beta^b, 1, 1, \beta^{b\beta}),
\end{aligned}$$

which implies that $\beta^b = \beta^{b\beta} = \beta$ and $\gamma^c = \gamma^{c\gamma} = \gamma$.

We have then,

$$\begin{aligned}\beta^c &= (\alpha^\beta, \gamma^{b\beta}, \alpha, \gamma^b)\\ \gamma^b &= (\beta^\alpha, 1, 1, \beta^c)\\ \gamma^{b\beta} &= (\beta, 1, 1, \beta^{c\gamma})\\ \beta^{c\gamma} &= (\alpha, \gamma^{b\beta}, \alpha, \gamma^{b\beta}).\end{aligned}$$

The last two equalities imply that $\beta^{c\gamma} = \beta$ and $\gamma^{b\beta} = \gamma$, which implies that $\beta^c = \beta^\gamma$ and $\gamma^b = \gamma^\beta$. This agrees with the first two equalities. $\square$

Proposition 4.5. *The group $\widetilde{\mathcal{D}}$ is level-transitive. The subgroup $\mathcal{D} < \widetilde{\mathcal{D}}$ is normal.*

Proof. The first statement follows from the fact that $\widetilde{\mathcal{D}}$ is transitive on the first level of the tree and that it is recurrent (i.e., restriction of the stabilizer of a vertex onto the corresponding sub-tree is an epimorphism), which is checked directly.

Normality follows from the relations that we have proved above. $\square$

We say that two sequences $w_1, w_2 \in \mathsf{X}^\omega$ are *cofinal* if they are of the form $w_1 = v_1 w, w_2 = v_2 w$ for some infinite word $w \in \mathsf{X}^\omega$ and two finite words $v_1, v_2 \in \mathsf{X}^*$ of equal length.

Let us call two sequences $w_1, w_2 \in \mathsf{X}^\omega$ *conjugate* if the groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$.

Proposition 4.6. *Let H be the group generated by the following automorphisms of the binary tree X^**

$$a = \sigma, b = (a, c), c = (b, b).$$

If $w_1, w_2 \in \mathsf{X}^\omega$ belong to one H-orbit, then the groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$ (and, in particular, are isomorphic).

Proof. Consider the map $(\mathsf{X} \times \mathsf{X})^* \to \mathsf{X}^*$, acting by

$$(x_1, y_1)(x_2, y_2) \ldots (x_n, y_n) \mapsto y_1 y_2 \ldots y_n.$$

Then the action of $\widetilde{\mathcal{D}}$ on $(\mathsf{X} \times \mathsf{X})^*$ agrees with this map (i.e., this map defines an imprimitivity system) so that the map induces an epimorphism of $\widetilde{\mathcal{D}}$ onto the group of automorphisms of X^* equal to the action of $\widetilde{\mathcal{D}}$ on the second coordinates of the letters. The group $\mathcal{D}$ belongs to the kernel of this epimorphism, since its elements do not change the second coordinates of the elements of $(\mathsf{X} \times \mathsf{X})^*$. It is easy to see that the generators a, b, c are mapped to the generators a, b, c of H, respectively.

If $w_2 = g(w_1)$ for some $g \in H$, then for any preimage g' of g in $\widetilde{\mathcal{D}}$ we have $g'(\mathsf{T}_{w_1}) = \mathsf{T}_{g(w_1)}$, where T_w, as before, denotes the subtree of $\mathsf{T} = (\mathsf{X} \times \mathsf{X})^*$ equal to union of the paths on whose second coordinates the word w is read.

Then the element g' conjugates the action of $\mathcal{D}$ on T_{w_1} with the action of $\mathcal{D}$ on T_{w_2}, which implies by Proposition 4.1 that $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$. $\square$

4.4. **Properties of the group H.** Recall that H is the quotient of the group $\widetilde{\mathcal{D}}$ by the kernel of its action on the second coordinates of the letters of $\mathsf{X} \times \mathsf{X}$ (see Proposition 4.6) and is generated by the automorphisms

$$a = \sigma, \quad b = (a, c), \quad c = (b, b).$$

We will show later that the kernel coincides with $\mathcal{D}$, so that H is the quotient $\widetilde{\mathcal{D}}/\mathcal{D}$.

Proposition 4.7. *The group H is given by the presentation*

$$\left\langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ba)^4 = (bc)^4 = 1 \right\rangle$$

and is isomorphic to the group of the symmetries of a square lattice, where a, b, c are the reflections $(x, y) \mapsto (x, -y)$, $(x, y) \mapsto (y, x)$ and $(x, y) \mapsto (1 - x, y)$ with respect to sides of the fundamental triangle with the vertices $(0, 0), (1/2, 1/2), (1/2, 0)$.

Proof. We have the following relations in H:

$$\begin{aligned} a^2 &= 1, \\ b^2 &= (a^2, c^2), \\ c^2 &= (b^2, b^2), \end{aligned}$$

hence $a^2 = b^2 = c^2 = 1$. Consequently, $(ac)^2 = (b^2, b^2) = 1$, $(ba)^4 = ((ac)^2, (ca)^2) = 1$ and $(bc)^4 = ((ab)^4, (cb)^4) = 1$. Consequently, the generators of H satisfy all the relations of the presentation, hence H is a quotient of the group of symmetries of a square lattice.

Consider the subgroup of H generated by ca, cb. It is the image of the subgroup of orientation preserving automorphism of the lattice. The element $ca = ac$ is the transformation $(1 - x, -y)$, i.e., rotation around $(1/2, 0)$ by π. The element cb is the transformation $(1 - y, x)$, i.e., rotation around $(1/2, 1/2)$ by $\pi/2$.

We have

$$\begin{aligned} ca &= \sigma(b, b) \\ cb &= (ba, bc). \end{aligned}$$

Passing to the basis $0, 1 \cdot b$ of the bimodule $\mathsf{X} \cdot H$, we get recursion

$$\begin{aligned} ca &= \sigma \\ cb &= (ba, cb) = (bc \cdot ca, cb). \end{aligned}$$

Let us prove that the group given by this recursion is isomorphic to the group of orientation preserving automorphisms of the square lattice (compare with [2]) with the same interpretation of the transformations ba and cb. This will imply that H is the group of symmetries of the square lattice.

Denote $ca = A$ and $cb = B$. Then the above recursion is written

$$\begin{aligned} A &= \sigma \\ B &= (B^{-1}A, B). \end{aligned}$$

We see that A is of order 2 and $(AB)^2 = (A, B^{-1}AB)$, hence AB and B are of order 4. Consider now the subgroup generated by B^2A and BAB.

We have

$$\begin{aligned} B^2A &= \sigma\left(B^2, B^{-1}AB^{-1}A\right), \\ BAB &= \sigma\left(A, B^{-1}AB\right). \end{aligned}$$

The commutator of these elements is

$$AB^2 \cdot B^{-1}AB^{-1} \cdot B^2A \cdot BAB = ABABABAB = 1,$$

hence the group $\left\langle B^2A, BAB \right\rangle$ is abelian. The first level stabilizer in this group is generated by $B^2A \cdot BAB$ and $B^2A \cdot B^{-1}AB^{-1}$, which satisfy

$$\begin{aligned} B^2ABAB &= \left(B^{-1}AB^{-1}, BAB\right) \\ B^2AB^{-1}AB^{-1} &= \left(B^{-1}AB^{-1}AB^{-1}AB, B^2A\right) = \left(AB^2, B^2A\right). \end{aligned}$$

We see that the subgroup $\langle B^2A, BAB\rangle$ of H is semi-invariant, with the associated virtual endomorphism mapping $B^2A\cdot BAB$ to BAB and $B^2A\cdot(BAB)^{-1}$ to B^2A (if we take the second coordinate of the wreath recursion), i.e., the virtual endomorphism is given by the matrix

$$\left(\begin{array}{cc} 1/2 & -1/2 \\ 1/2 & 1/2 \end{array}\right).$$

The eigenvalues of this matrix are $\frac{1+i}{2}$ and $\frac{1-i}{2}$, hence $\langle B^2A, BAB\rangle \cong \mathbb{Z}^2$ by Proposition 2.9.2 of [9] (see also [11]).

Conjugation of this group by A is the automorphism $B^2A \mapsto AB^2 = (B^2A)^{-1}$, $BAB \mapsto ABABA = (B^2A)^{-1}\cdot B^{-1}AB^{-1}\cdot B^2A = B^{-1}AB^{-1}$, hence it is the automorphism mapping every element to its inverse.

Conjugation by B acts on the generators by the rule $B^2A \mapsto BAB$ and $BAB \mapsto AB^2 = (B^2A)^{-1}$, hence it is the automorphism given by the matrix $\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)$. This implies that $\langle A, B\rangle$ is isomorphic to the group of orientation preserving automorphisms of the square lattice, where A rotates the lattice by π and B is a rotation by $\pi/2$. The proof of the proposition follows directly. □

Proposition 4.8. *The set of the elements of $\widetilde{\mathcal{D}}$ acting trivially on the second coordinates of the letters coincides with $\mathcal{D}$. Consequently, the quotient $\widetilde{\mathcal{D}}/\mathcal{D}$ is isomorphic to H.*

Proof. Let us write the elements of $\widetilde{\mathcal{D}}$ corresponding to the relators of the presentation of H.

We have $[\alpha, a] = [b, \beta] = 1$, which implies that the elements a, b and c are of order 2.

The elements b and β commute, hence

$$(ac)^2 = (ca)^2 = (\beta, \beta, \beta, \beta) = (\alpha\gamma)^2 = (\gamma\alpha)^2$$

The element α commutes with a and c, hence

$$(ba)^4 = \left((ac)^2, (ac)^2, (ca)^2, (ca)^2\right) = \left((\alpha\gamma)^2, (\alpha\gamma)^2, (\alpha\gamma)^2, (\alpha\gamma)^2\right) = (\alpha\beta)^4$$

Similarly $(ab)^4 = (\alpha\beta)^4$.

Again, the elements a, b commute with the elements α, β, so that

$$\begin{aligned} (bc)^4 &= \left((a\alpha b\beta)^4, (a\alpha b\beta)^4, (cb)^4, (cb)^4\right) = \\ &\left((ab)^4(\alpha\beta)^4, (ab)^4(\alpha\beta)^4, (cb)^4(cb)^4\right) = \\ &\left((\alpha\beta)^8, (\alpha\beta)^8, (cb)^4, (cb)^4\right) = (1, 1, (cb)^4, (cb)^4), \end{aligned}$$

hence $(bc)^4 = 1$.

We see that the relators of the presentation from Proposition 4.7 belong to $\mathcal{D}$. The group $\mathcal{D}$ is normal in $\widetilde{\mathcal{D}}$, therefore, if a group word in $a, b, c \in \widetilde{\mathcal{D}}$ is a relation in H, then it is an element of $\mathcal{D}$. □

The following proposition is proved by direct computations, or as Proposition 4.2.

Proposition 4.9. *The group H is contracting with the nucleus equal to the set*

$$\langle a, b\rangle \cup \langle a, c\rangle \cup \langle b, c\rangle.$$

We have

$$1, \quad a = \sigma, \quad b = (a, c), \quad c = (b, b),$$

$$ab = \sigma(a, c), \quad ba = \sigma(c, a), \quad ca = \sigma(b, b),$$
$$cb = (ba, bc), \quad bc = (ab, cb), \quad b^a = (c, a),$$
$$a^b = \sigma(ca, ac), \quad (ab)^2 = (ca, ca),$$
$$c^b = (b^a, b^c), \quad b^c = (a^b, c^b), \quad (bc)^2 = ((ab)^2, (bc)^2).$$

Proposition 4.10. *The orbits of H on X^ω coincide with the cofinality classes.*

Proof. The orbits are unions of the cofinality classes, since the group H is recurrent (i.e., its left action on the self-similarity bimodule is transitive, see [9]...). Let us prove that every H-orbit belongs to a cofinality class.

If w_1 and w_2 belong to one H-orbit, then there exists an element of the nucleus g and n such that $g(s^n(w_1)) = s^n(w_2)$. Hence it is sufficient to prove that every element of the nucleus preserves the cofinality classes of sequences.

Consider the following partition of the nucleus into "levels".

$$\begin{array}{rcl}
\mathcal{N}_5 & = & \{c^b, b^c, (bc)^2\} \\
\mathcal{N}_4 & = & \{cb, bc, b^a, a^b, (ab)^2\} \\
\mathcal{N}_3 & = & \{ab, ba, ca\} \\
\mathcal{N}_2 & = & \{b, c\} \\
\mathcal{N}_1 & = & \{a\} \\
\mathcal{N}_0 & = & \{1\}.
\end{array}$$

It is easy to check that for every $g \in \mathcal{N}$ and $x \in \mathsf{X}$ either $g|_x$ belongs to a lower level, or it belongs to the same level and $g(x) = x$. Moreover, if $g(x) \neq x$, then either $g = a$, or g belongs to $\mathcal{N}_3$ (for $g = ca, ba$ or ab) or to $\mathcal{N}_4$ (for $g = a^b$). Consequently, every element of the nucleus changes at most 3 letters of an infinite sequence and hence preserves the cofinality classes. □

4.5. **Contraction of $\widetilde{\mathcal{D}}$.**

Proposition 4.11. *The group $\widetilde{\mathcal{D}}$ is contracting.*

The proposition follows from the next general argument (the fact that the generators of $\widetilde{\mathcal{D}}$ are finite-state is easy to check).

Proposition 4.12. *Suppose that we have a self-similar group $\widetilde{G}$ over the alphabet $\mathsf{X} \times \mathsf{Y}$. Let $G \leq \widetilde{G}$ be the subgroup acting trivially on Y, i.e., the subgroup of the elements $g \in \widetilde{G}$ such that $g((x_1, y_1) \ldots (x_n, y_n) = (z_1, y_1) \ldots (z_n, y_n)$. Let $H = \widetilde{G}/G$ be the quotient describing the action of G on X^*. Suppose that the groups G and H are contracting and the group $\widetilde{G}$ is finite-state. Then the group $\widetilde{G}$ is contracting.*

Proof. Let $\mathcal{N}_H$ and $\mathcal{N}_G$ be the nuclei of the groups H and G, respectively. Let $\widetilde{\mathcal{N}_H}$ be a finite state-closed subset of $\widetilde{G}$ whose image in H contains $\mathcal{N}_H$. Such a set exists, since $\widetilde{G}$ is finite-state.

Let $g \in \widetilde{G}$ be an arbitrary element. There exists n_0 such that for every word $v \in (\mathsf{X} \times \mathsf{Y})^*$ of length more than n_0 the image of the restriction $g|_v$ in H belongs to $\mathcal{N}_H$. Hence $g|_v$ can be written in the form $g' \cdot h$, where $g' \in G$ and $h \in \widetilde{\mathcal{N}_H}$. Then there exists n_1 such that for every word v of length more than n_1 the restriction $g' \cdot h|_v$ belongs to $\mathcal{N}_G \cdot \widetilde{\mathcal{N}_H}$. Consequently, the group $\widetilde{G}$ is contracting with the nucleus a subset of $\mathcal{N}_G \cdot \widetilde{\mathcal{N}_H}$. □

5. Isomorphism classes of $\mathcal{D}_w$

5.1. Complete description of the conjugacy classes.

Theorem 5.1. *Two groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate if and only if w_1 and w_2 are cofinal.*

Proof. We have proved this theorem in one direction (see Proposition 4.6).

Suppose that $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate. We have to prove that w_1 and w_2 are cofinal. It is sufficient to prove that the sequences $s^n(w_1)$ and $s^n(w_2)$ are cofinal for some n, or that they belong to one H-orbit.

The groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate if and only if there exists a generating set h_0, h_1, h_2 of $\mathcal{D}_{w_1}$ such that the triple (h_0, h_1, h_2) is simultaneously conjugate to $(\alpha_{w_2}, \beta_{w_2}, \gamma_{w_2})$. Moreover, $w_2 = W(h_0, h_1, h_2)$ is uniquely determined by (h_0, h_1, h_2).

Lemma 5.2. *Let (h_0, h_1, h_2) be a t-triple of elements of $\mathcal{D}_{w_1}$ and denote $w_2 = W(h_0, h_1, h_2)$. Then it is of the form*

$$\begin{aligned} h_0 &= \sigma(g, g^{-1}), \\ h_1 &= (g_0, g_2), \textit{ or } (g_2, g_0), \\ h_2 &= (g_1, 1), \textit{ or } (1, g_1). \end{aligned}$$

for some t-triple (g_0, g_1, g_2).

Let us denote

$$s(h_0, h_1, h_2) = \begin{cases} (g_0, g_1, g_2^g) & \text{if } h_1 = (g_0, g_2) \text{ and } h_2 = (g_1, 1), \\ (g_0, g_1, g_2^{g^{-1}}) & \text{if } h_1 = (g_2, g_0) \text{ and } h_2 = (1, g_1), \\ (g_0^{g^{-1}}, g_1, g_2) & \text{if } h_1 = (g_0, g_2) \text{ and } h_2 = (1, g_1), \\ (g_0^g, g_1, g_2) & \text{if } h_1 = (g_2, g_0) \text{ and } h_2 = (g_1, 1). \end{cases}$$

The first two cases (when $s(h_0, h_1, h_2) = (g_0, g_1, g_2^{g^{\pm 1}})$) happen when the first letter of w_2 is 0. The other two cases (when $s(h_0, h_1, h_2) = (g_0^{g^{\pm 1}}, g_1, g_2)$) take place when the first letter of w_2 is 1.

In all cases $s(h_0, h_1, h_2)$ is a t-triple of elements of $\mathcal{D}_{s(w_1)}$ and

$$W(s(h_0, h_1, h_2)) = s(w_2).$$

Proof. Follows directly from the proof of Proposition 3.1. $\square$

For every $w \in \mathsf{X}^\omega$ and every $g \in \mathcal{D}_w$ denote by $\ell(g)$ the minimal number of appearances of the generators β_w and γ_w in a product equal to g. It is easy to prove that ℓ is a good length function, i.e., that it satisfies the triangle inequality and that for every R the number of elements g such that $\ell(g) \le R$ is finite (the last follows from the fact that α_w is of order 2).

It follows directly from the recurrent definition of the generators of $\mathcal{D}_w$ that if $h \in \mathcal{D}_w$ and $h = \sigma(h_0, h_1)$, or $h = (h_0, h_1)$, then $\ell(h_0) + \ell(h_1) \le \ell(h)$.

Let us denote $w = s^n(w_1)$ and $w' = s^n(w_2) = W(h_0, h_1, h_2)$.

We will write the generators $\alpha_w, \beta_w, \gamma_w$ of $\mathcal{D}_w$ just α, β, γ, since in every case it will be clear what w is considered. If we consider elements of a group $\mathcal{D}_w$ then the *current letter* is the first letter of the sequence w. Decomposition of γ under the wreath recursion will depend on the current letter.

Lemma 5.3. *For every t-triple $(\widetilde{h}_0, \widetilde{h}_1, \widetilde{h}_2)$ of elements of $\mathcal{D}_{w_1}$ there exists n_0 such that the triple $(h_0, h_1, h_2) = s^n(\widetilde{h}_0, \widetilde{h}_1, \widetilde{h}_2)$ satisfies for all $n \geq n_0$ the following conditions, where i is the first letter of $s^n(w_1)$.*

$$\begin{array}{lcl} h_1 & \in & \left\{ \begin{array}{ll} \{\beta, \beta^\gamma, \beta^{\gamma\beta}, \beta^{\gamma\beta\gamma}\} & \text{if } i = 1 \\ \{\beta, \beta^\gamma\} & \text{if } i = 0 \end{array} \right. \\ h_2 & \in & \left\{ \begin{array}{ll} \{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}, \gamma^\alpha, \gamma^{\beta\alpha}, \gamma^{\beta\gamma\alpha}, \gamma^{\beta\gamma\beta\alpha}\} & \text{if } i = 1 \\ \{\gamma, \gamma^{\beta^\alpha}, \gamma^{\beta^\alpha\gamma}, \gamma^{\beta^\alpha\gamma\beta^\alpha}, \gamma^\alpha, \gamma^{\alpha\beta}, \gamma^{\alpha\beta\gamma^\alpha}, \gamma^{\alpha\beta\gamma^\alpha\beta}\} & \text{if } i = 0 \end{array} \right. \end{array}$$

Proof. In conditions of the previous lemma, either

$$\begin{array}{rcl} \ell(g_0) + \ell(g_1) + \ell(g_2^{g^{\pm 1}}) & \leq & \ell(g_0) + \ell(g_1) + \ell(g_2) + 2\ell(g) \\ & \leq & \ell(h_0) + \ell(h_1) + \ell(h_2), \end{array}$$

or

$$\begin{array}{rcl} \ell(g_0^{g^{\pm 1}}) + \ell(g_1) + \ell(g_2) & \leq & \ell(g_0) + 2\ell(g) + \ell(g_1) + \ell(g_2) \\ & \leq & \ell(h_0) + \ell(h_1) + \ell(h_2). \end{array}$$

We see that the shift s does not increase the sum $\ell(h_0) + \ell(h_1) + \ell(h_2)$. Hence, it becomes constant for $(h_0, h_1, h_2) = s^n(\widetilde{h}_0, \widetilde{h}_1, \widetilde{h}_2)$ starting from some n.

After that we must have equalities, so that

$$\ell(h_0) = 2\ell(g), \quad \ell(h_1) = \ell(g_0) + \ell(g_1), \quad \ell(h_2) = \ell(g_1)$$

and

$$\ell(g_2^{g^{\pm 1}}) = 2\ell(g) + \ell(g_2),$$

or

$$\ell(g_0^{g^{\pm 1}}) = 2\ell(g) + \ell(g_0),$$

accordingly to the cases in Lemma 5.2.

The element h_2 belongs to the first level stabilizer, which is generated by $\beta, \gamma, \beta^\alpha$ and γ^α. One coordinate of each of these generators has ℓ-length 1 (and equal either to β or to γ) and the other has ℓ-length 0 (and equal either to 1 or to α). So, equality $\ell(h_2) = \ell(g_1)$ means that all coordinates of ℓ-length one are collected in one place and no cancelling happens. In particular, this implies that $g_1 \in \langle \beta, \gamma \rangle$. Replacing (h_0, h_1, h_2) by $s(h_0, h_1, h_2)$, if necessary, we may assume that $h_1 \in \langle \beta, \gamma \rangle$.

Note that, by definition, the element h_1 must be conjugate to β in the full automorphism group of the binary tree. Looking through the elements of the group $\langle \beta, \gamma \rangle \cong D_8$, we can easily check that we have only four possibilities: $\beta, \beta^\gamma, \beta^{\gamma\beta}, \beta^{\gamma\beta\gamma}$.

If the current letter of h_1 is equal to 0, then

$$\begin{array}{rcl} \beta^\gamma & = & (\alpha^\beta, \gamma) \\ \beta^{\gamma\beta} & = & (\alpha^{\beta\alpha}, \gamma^\gamma), \end{array}$$

and we see that additional cancelling takes place, so only the cases $h_1 = \beta = (\alpha, \gamma)$ and $h_1 = \beta^\gamma = (\alpha^\beta, \gamma)$ are possible.

If the current letter is equal to 1, then we get

$$\begin{array}{rcl} \beta^\gamma & = & (\alpha, \gamma^\beta) \\ \beta^{\gamma\beta} & = & (\alpha, \gamma^{\beta\gamma}) \\ \beta^{\gamma\beta\gamma} & = & (\alpha, \gamma^{\beta\gamma\beta}). \end{array}$$

The same conditions are applied to g_1. But h_2 is of the form $(1, g_1)$ or $(g_1, 1)$, hence

$$h_2 \in \left\{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}, \gamma^\alpha, \gamma^{\beta\alpha}, \gamma^{\beta\gamma\alpha}, \gamma^{\beta\gamma\beta\alpha}\right\},$$

if the current letter of h_2 is 1, and

$$h_2 \in \left\{\gamma^\alpha, \gamma^{\alpha\beta}, \gamma^{\alpha\beta\gamma^\alpha}, \gamma^{\alpha\beta\gamma^\alpha\beta}, \gamma, \gamma^{\beta^\alpha}, \gamma^{\beta^\alpha\gamma}, \gamma^{\beta^\alpha\gamma\beta^\alpha}\right\},$$

if it is 0. $\square$

Lemma 5.4. *Suppose that $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate and infinitely many digits of w_1 and w_2 are zeros. Then there exists n and a t-triple (h_0, h_1, h_2) of generators of $\mathcal{D}_{s^n(w_1)}$ such that $W(h_0, h_1, h_2) = s^n(w_2)$ and*

$$\begin{aligned} h_0 &\in \left\{\alpha, \alpha^\beta, \alpha^\gamma\right\} \\ h_1 &\in \left\{\beta, \beta^\gamma, \beta^{\gamma\beta}, \beta^{\gamma\beta\gamma}\right\} \\ h_2 &\in \left\{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}\right\} \end{aligned}$$

Proof. Let (h_0, h_1, h_2) be a t-triple with the smallest possible value of $\ell(h_0)+\ell(h_1)+\ell(h_2)$ such that it generates $\mathcal{D}_{s^n(w_1)}$ and $W(h_0, h_1, h_2) = s^n(w_2)$ for some n. We will call such t-triples *minimal*. We have shown in the proof of the previous lemma that if (h_0, h_1, h_2) is minimal, then $s^k(h_0, h_1, h_2)$ is minimal for every k.

We may assume hence that (h_0, h_1, h_2) in the conditions of our Lemma is minimal, satisfies the conditions of Lemma 5.3, and that the first letter of w_2 is zero. Then the first element of $s(h_0, h_1, h_2)$ is either α or α^β. Let us pass then to $s(h_0, h_1, h_2)$ and thus assume that, in addition to the conditions on h_1 and h_2 from Lemma 5.3, we have $h_0 \in \left\{\alpha, \alpha^\beta\right\}$ (we do not assume any more that w_1 starts with 0).

Let us denote $(h_0', h_1', h_2') = s(h_0, h_1, h_2)$. Suppose that $h_0 = \alpha$. If the first letter of w_1 is 0, then we get $h_0' \in \left\{\alpha, \alpha^\beta\right\}$, $h_2' = \gamma$ and the condition on h_1' from Lemma 5.3.

If the first letter of w_1 is 1, then we get $h_0' = \alpha$, the condition on h_1' from Lemma 5.3 and $h_2' \in \left\{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}\right\}$.

We see that the lemma is true if $h_0 = \alpha$.

Suppose now that $h_0 = \alpha^\beta = \sigma(\gamma\alpha, \alpha\gamma)$. We have then $h_1 = (g_0, g_2)$ (see Lemma 5.3) and $h_2 = (g_1, 1)$ or $h_2 = (1, g_1)$ for a t-triple (g_0, g_1, g_2).

We consider now four cases depending on the first letters of w_1 and w_2.

Suppose that w_1 and w_2 both start with zero. Then $h_0' = g_0 \in \left\{\alpha, \alpha^\beta\right\}$, $h_2' = g_2^{\gamma\alpha} = \gamma^{\gamma\alpha}$, which contradicts with the minimality (a cancelling in projections happens). Therefore this case is impossible.

Suppose that w_1 starts with 0 and w_2 with 1. Then $h_0' = g_0^{\alpha\gamma} \in \left\{\alpha^\gamma, \alpha^{\beta\alpha\gamma}\right\}$, $h_2' = g_2 = \gamma$. Replacing (h_0', h_1', h_2') by $({h_0'}^\gamma, {h_1'}^\gamma, {h_2'}^\gamma)$ must not decrease the total ℓ-length of the triple, therefore h_1' is either β, or $\beta^{\gamma\beta}$. In these cases conjugation by γ does not change the total length, hence we get a minimal t-triple

$$\left({h_0'}^\gamma, {h_1'}^\gamma, {h_2'}^\gamma\right) = \left(\alpha \text{ or } \alpha^{\beta\alpha}, \beta^\gamma \text{ or } \beta^{\gamma\beta\gamma}, \gamma\right).$$

If w_1 starts with 1 and w_2 with 0, then $h_0' = g_0 = \alpha$ and $h_2' = g_2^{\gamma\alpha}$. We know that $g_2 \in \left\{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}\right\}$, and since no cancelling in projections may happen, we get

$$h_2' \in \left\{\gamma^{\beta\gamma\alpha}, \gamma^{\beta\gamma\beta\alpha}\right\},$$

and we get a triple, satisfying conditions of the lemma.

If both w_1 and w_2 start with 1, then we have $h_0' = g_0^{\alpha\gamma} = \alpha^\gamma$ and $h_2' = g_2 \in \left\{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}\right\}$, which also satisfies the conditions of the lemma.

So, the only cases which still have to be considered are the triples of the form

$$\begin{array}{rcl} h_0' &=& \alpha^{\beta\alpha} \\ h_1' &\in& \left\{\beta^\gamma, \beta^{\gamma\beta\gamma}\right\} \\ h_2' &=& \gamma \end{array}$$

and

$$\begin{array}{rcl} h_0' &=& \alpha \\ h_1' &\in& \left\{\beta, \beta^\gamma, \beta^{\gamma\beta}, \beta^{\gamma\beta\gamma}\right\} \\ h_2' &\in& \left\{\gamma^{\beta\gamma\alpha}, \gamma^{\beta\gamma\beta\alpha}\right\}. \end{array}$$

Let us consider the first case. If the current letter for the elements h_i' is 0, then $h_1' = \beta^\gamma = \left(\alpha^\beta, \gamma\right)$, $\alpha^{\beta\alpha} = \sigma\left(\alpha\gamma, \gamma\alpha\right)$, $h_2' = \gamma = (\beta, 1)$ and $h_0' = \sigma\left(\alpha\gamma, \gamma\alpha\right)$. Then $s(h_0', h_1', h_2') = \left(\alpha^\beta, \beta, \gamma^{\alpha\gamma}\right)$, which contradicts with minimality, since $\gamma^{\alpha\gamma} = \gamma^\alpha$.

Suppose now that the current letter is 1. Then h_1' is either $\left(\alpha, \gamma^\beta\right)$, or $\left(\alpha, \gamma^{\beta\gamma\beta}\right)$. Since $h_0' = \sigma\left(\alpha\gamma, \gamma\alpha\right)$, we have $s(h_0', h_1', h_2') = \left(\alpha^\gamma, \beta, \gamma^\beta\right)$ or $\left(\alpha^\gamma, \beta, \gamma^{\beta\gamma\beta}\right)$. We have used $\alpha^\gamma = \sigma(\beta, \beta)$, hence $\alpha^{\gamma\alpha} = \alpha^\gamma$.

Let us consider now the second case. It is possible only if the first letter of $s(w)$ is 1. Then $s(h_0', h_1', h_2')$ is of the form

$$\left(\alpha, \left\{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}\right\}, \left\{\beta^{\gamma\beta}, \beta^{\gamma\beta\gamma}\right\}\right)$$

Hence we see that in any case we get a triple of the form described in the lemma. □

Lemma 5.5. *Suppose that $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are conjugate and each of the sequences w_1 and w_2 has infinitely many zeros. Then there exists n and a t-triple (h_0, h_1, h_2) of generators of $\mathcal{D}_{s^n(w_1)}$ such that $W(h_0, h_1, h_2) = s^n(w_2)$ and (h_0, h_1, h_2) is either (α, β, γ) or $(\alpha, \beta, \gamma^\beta)$.*

Proof. By Lemma 5.4, there exists n and a t-triple (h_0, h_1, h_2) of generators of $\mathcal{D}_{s^n(w_1)}$ such that $W(h_0, h_1, h_2) = s^n(w_2)$ and

$$\begin{array}{rcl} h_0 &\in& \left\{\alpha, \alpha^\beta, \alpha^\gamma\right\} \\ h_1 &\in& \left\{\beta, \beta^\gamma, \beta^{\gamma\beta}, \beta^{\gamma\beta\gamma}\right\} \\ h_2 &\in& \left\{\gamma, \gamma^\beta, \gamma^{\beta\gamma}, \gamma^{\beta\gamma\beta}\right\} \end{array}$$

Suppose that the first letter of $s^n(w_2)$ is 1. If $h_0 = \alpha$, then $(h_0', h_1', h_2') = s(h_0, h_1, h_2)$ is a t-triple of the same form and $h_0' = \alpha$.

If $h_0 = \alpha^\beta = \sigma(\gamma\alpha, \alpha\gamma)$, then the triple $(h_0', h_1', h_2') = s(h_0, h_1, h_2)$ again satisfies the conditions of Lemma 5.4. with $h_0' = \alpha^\gamma$.

If $h_0 = \alpha^\gamma = \sigma(\beta, \beta)$, then we similarly get a triple satisfying the conditions of Lemma 5.4 with $h_0' = \alpha^\beta$.

Consider now the case when the first digit of $s^n(w_1)$ is 0. Then we get the following cases (see Lemma 5.3)

Case A: a triple of the form

$$\left(\alpha, \left\{\beta, \beta^\gamma\right\}, \gamma\right).$$

Case B: a triple of the form

$$\left(\alpha^\beta, \{\beta, \beta^\gamma\}, \gamma\right).$$

Case C: a triple of the form

$$\left(\alpha^\gamma, \{\beta, \beta^\gamma\}, \gamma\right).$$

Case A is mapped by the shift to (α, β, γ) or to $(\alpha^\beta, \beta, \gamma)$, which is conjugate to $(\alpha, \beta, \gamma^\beta)$.

Case B contradicts minimality, since we get the third coordinate of the next triple equal to $\gamma^{\gamma\alpha}$.

Case C is mapped to $(\alpha, \beta, \gamma^\beta)$ or to $(\alpha^\beta, \beta, \gamma^\beta)$, the last case contradicting minimality.

This finishes the proof for the case when both sequences have infinitely many zeros. □

Now let us consider the case when one of the sequences has only a finite number of zeros. We may consider only the case $w_1 = 111\ldots$. We know that then β and γ commute, hence by Lemma 5.3, we may assume that $h_1 = \beta$ and $h_2 \in \{\gamma, \gamma^\alpha\}$.

We may assume that w_2 has infinitely many zeros, since otherwise there is nothing to prove. We may assume, hence that the first letter of w_2 is 0. Then the triple $s(h_0, h_1, h_2)$ is of the form

$$(\alpha, \beta, \gamma), \text{ or } (\alpha, \beta, \gamma^\alpha),$$

which finishes the proof. □

5.2. **Relations in $\mathcal{D}_w$.** Consider the following wreath recursions $\psi_i : \mathcal{D} \to C_2 \wr \mathcal{D}$:

$$\begin{array}{ll} \psi_0(\alpha) = \sigma, & \psi_1(\alpha) = \sigma \\ \psi_0(\beta) = (\alpha, \gamma), & \psi_1(\beta) = (\alpha, \gamma) \\ \psi_0(\gamma) = (\beta, 1), & \psi_1(\gamma) = (1, \beta). \end{array}$$

Let Ψ_0 and Ψ_1 be the respective $\mathcal{D}$-bimodules. Denote by $\psi_{x_1x_2\ldots x_n}$ be the wreath recursion associated with the tensor product $\Psi_{x_1}\otimes\Psi_{x_2}\otimes\cdots\otimes\Psi_{x_n}$. It follows directly from the definitions that the direct sum $\Psi_0\oplus\Psi_1$ is the self-similarity bimodule of the group $\mathcal{D}$ and that the wreath recursion $\psi_{x_1x_2\ldots x_n}$ is the nth level wreath recursion defining the automorphisms α_w, β_w, γ_w for w starting with $x_1x_2\ldots x_n$.

Denote by $\mathcal{E}_{x_1x_2\ldots x_n}$ the kernel of $\psi_{x_1x_2\ldots x_n}$, i.e., the kernel of the left action of Γ on the bimodule $\Psi_{x_1} \otimes \Psi_{x_2} \otimes \cdots \otimes \Psi_{x_n}$.

We obviously have $\mathcal{E}_{x_1x_2\ldots x_n} \le \mathcal{E}_{x_1x_2\ldots x_nx_{n+1}}$ for all $x_1x_2\ldots x_{n+1} \in \mathsf{X}^*$. We obtain a rooted binary tree of normal subgroups of $\mathcal{D}$.

Let us denote for $x_1x_2\ldots \in \mathsf{X}^\omega$

$$\mathcal{E}_{x_1x_2\ldots} = \bigcup_{n\ge 0} \mathcal{E}_{x_1x_2\ldots x_n}.$$

Proposition 5.6. *Suppose that the sequence $w = x_1x_2\ldots$ has infinitely many zeros. Then $\mathcal{E}_w$ is the kernel $\mathcal{K}_w$ of the natural epimorphism $\mathcal{D} \to \mathcal{D}_w$.*

Proof. The proof is the same as the proof of an analogous statement about self-similar groups (see Proposition 2.13.2 of [9]).

It is obvious that $\mathcal{E}_w$ belongs to the kernel of the epimorphism. Suppose now that $g \in \mathcal{D}$ belongs to $\mathcal{K}_w$. By Proposition 4.2 there exists n such that for every $a_1a_2\ldots a_n \in \mathsf{X}^*$ we have

$$g \cdot (a_1, x_1)(a_2, x_2)\ldots(a_n, x_n) = (a_1, x_1)(a_2, x_2)\ldots(a_n, x_n) \cdot h$$

for $h \in \mathcal{N}$. But the intersection of $\mathcal{N}$ with $\mathcal{K}_{s^n(w)}$ is trivial, if $s^n(w) \neq 11\ldots$ by Proposition 4.3. □

If f is an automorphism of a group G, then by $[f]$ we denote the permutational G-bimodule consisting of expressions of the form $f \cdot g$ with the right action

$$(f \cdot g) \cdot h = f \cdot gh$$

and the left action

$$h \cdot (f \cdot g) = f \cdot h^f g.$$

It is easy to see that $[f]$ is a covering G-bimodule with transitive left and right actions.

Let a, b, c be the automorphisms of $\mathcal{D}$ induced by conjugation from $\widetilde{\mathcal{D}}$.

Two G-bimodules Φ_1 and Φ_2 are *isomorphic* if there exists a bijection $F : \Phi_1 \to \Phi_2$ such that $F(g \cdot m \cdot h) = g \cdot F(m) \cdot h$ for all $g, h \in G$ and $m \in \Phi_1$.

Proposition 5.7. *We have the following isomorphisms of $\mathcal{D}$-bimodules:*

$$\begin{array}{ll} [a] \otimes \Psi_0 \cong \Psi_1, & [a] \otimes \Psi_1 \cong \Psi_0 \\ [b] \otimes \Psi_0 \cong \Psi_0 \otimes [a], & [b] \otimes \Psi_1 \cong \Psi_1 \otimes [c] \\ [c] \otimes \Psi_0 \cong \Psi_0 \otimes [b], & [c] \otimes \Psi_1 \cong \Psi_1 \otimes [b]. \end{array}$$

Proof. Follows directly from the equalities below, which are checked directly.

$$\begin{array}{ll} \psi_0\left(g^a\right) = \psi_1(g), & \psi_1\left(g^a\right) = \psi_0(g) \\ \psi_0\left(g^b\right) = \left(\psi_0(g)\right)^{a\alpha}, & \psi_1\left(g^b\right) = \left(\psi_1(g)\right)^{c} \\ \psi_0\left(g^c\right) = \left(\psi_0(g)\right)^{b\beta}, & \psi_1\left(g^c\right) = \left(\psi_1(g)\right)^{b}, \end{array}$$

where the automorphism on the right-hand sides acts on the wreath product $\mathfrak{S}(\mathsf{X}) \ltimes (\mathcal{D} \times \mathcal{D})$ by the diagonal action on $\mathcal{D} \times \mathcal{D}$. □

Corollary 5.8. *Let g be one of the symbols a, b, c. If $g \cdot x_1 x_2 \ldots x_n = y_1 y_2 \ldots y_n \cdot h$ in H for $h \in \{a, b, c, 1\}$, then*

$$[g] \otimes \Psi_{x_1} \otimes \Psi_{x_2} \otimes \cdots \otimes \Psi_{x_n} \cong \Psi_{y_1} \otimes \Psi_{y_2} \otimes \cdots \otimes \Psi_{y_n} \otimes [h].$$

Proof. The isomorphisms in Proposition 5.7 repeat the recursions defining H. □

The next proposition follows directly from Corollary 5.8.

Proposition 5.9. *If $g(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n$, then*

$$\left(\mathcal{E}_{x_1 x_2 \ldots x_n}\right)^{g^{-1}} = \mathcal{E}_{y_1 y_2 \ldots y_n}.$$

Let us denote, for $w = x_1 x_2 \ldots \in \mathsf{X}^\omega$, by G_w the quotient $\mathcal{D}/\mathcal{E}_{x_1 x_2 \ldots}$. If w has infinitely many zeros, then G_w is isomorphic to $\mathcal{D}_w$ (by Proposition 5.6). Otherwise it is isomorphic to an overgroup of the Grigorchuk group $\mathcal{D}_{111\ldots}$.

A group word g in a generating set of a group G is called an *inequality*, if $g \neq 1$.

Theorem 5.10. *Let $w \in \mathsf{X}^\omega$ be an arbitrary infinite sequence and let $A, B \subset \langle \alpha, \beta, \gamma \rangle = F$ be finite sets of relations and inequalities in G_w, i.e., such finite sets that $A \subset \mathcal{E}_w$ and $B \cap \mathcal{E}_w = \emptyset$.*

Then there exists n such that for any sequence $w' \in \mathsf{X}^\omega$ having a common beginning of length n with w the same relations and inequalities hold in $G_{w'}$.

Proof. Let $w = x_1 x_2 \ldots$. We can find n such that $A \subset \mathcal{E}_{x_1 x_2 \ldots x_n}$. Then for every $w' \in \mathsf{X}^\omega$ having a common beginning of length $\geq n$ with w we will have $A \subset \mathcal{E}_{w'}$.

For every $g \notin \mathcal{E}_w$ there exists n such that either the image of g in $\mathcal{D}_w$ acts non-trivially on the first n levels of the tree X^*, or it acts trivially on the tree X^*

but there exists n such that all coordinates of $\psi_{x_1\ldots x_n}(g)$ belong to $\mathcal{N}$, but some of them are non-trivial elements of $\mathcal{D}$. It is easy to see that all non-trivial elements of $\mathcal{N}$ do not belong to $\mathcal{E}_u$ for any $u \in \mathsf{X}^\omega$.

In both cases the same conditions will be also true for any sequence $w' \in \mathsf{X}^\omega$ with a common beginning of length at least n with w, hence $g \notin \mathcal{E}_{w'}$ for any such w'.

If we choose now a common n for all the elements of A and B, then it will satisfy the conditions of the theorem. $\square$

Corollary 5.11. *The map $w \mapsto (G_w, \alpha_w, \beta_w, \gamma_w)$ from X^ω to the set the space $\mathfrak{G}_3$ of 3-generated groups is continuous.*

Minimality of the set $\{G_w\}$ can be formulated now in the following terms.

Proposition 5.12. *If two sequences $w_1, w_2 \in \mathsf{X}^\omega$ belong to one H-orbit (i.e., are cofinal), then the groups G_{w_1} and G_{w_2} are isomorphic.*

For any two sequences $w_1, w_2 \in \mathsf{X}^\omega$ and for any finite sets A and B of relations and inequalities between the generators $\alpha_{w_1}, \beta_{w_1}, \gamma_{w_2}$ of G_{w_1} there exists a generating set h_0, h_1, h_2 of G_{w_2} such that the same relation and inequalities hold.

Proof. The first statement is a direct corollary of Proposition 5.9.

Let us prove the second statement. Let n be such that for any sequence w having a common beginning of length n with w_1 the relations and inequalities A, B hold for the generators $\alpha_w, \beta_w, \gamma_w$ of G_w. There exists, by level-transitivity of H, an element $h \in H$ such that $h(w_2)$ has a common beginning of length n with w_1. Then the relations and inequalities A and B hold for $\alpha_{h(w_2)}, \beta_{h(w_2)}, \gamma_{h(w_2)}$, hence they hold for $\alpha_{h(w_2)}^{h^{-1}}, \beta_{h(w_2)}^{h^{-1}}, \gamma_{h(w_2)}^{h^{-1}}$, which are generators of G_{w_2} by Proposition 5.9. $\square$

The next proposition shows that the groups $\mathcal{D}_w$ can be not distinguished by their action on finite portions of the rooted tree.

Proposition 5.13. *The closures of the groups $\mathcal{D}_w$ in $\mathrm{Aut}(\mathsf{X}^*)$ do not depend on w up to conjugacy in $\mathrm{Aut}(\mathsf{X}^*)$.*

Proof. The group H is level-transitive, since it is recurrent (i.e., the left action on the self-similarity bimodule is transitive). Hence, for any $w_1 = x_1x_2\ldots, w_2 = y_1y_2\ldots \in \mathsf{X}^\omega$ and $n \in \mathbb{N}$ there exists $h \in H$ such that $h(x_1x_2\ldots x_n) = y_1y_2\ldots y_n$.

Then it follows from Corollary 5.8 that for every $g \in \mathcal{D}$ the actions of $g_{w_1}^h$ and g_{w_2} on the first n levels of the binary tree are conjugate. Thus, the actions of the groups $\mathcal{D}_w$ on any finite portion of the binary tree are conjugate, so they have conjugate closures. $\square$

5.3. **$\mathcal{D}_w$ as branch groups.** Let us denote by L_w equal to the normal closure in $\mathcal{D}_w$ of the commutators $[\alpha, \beta]$ and $[\gamma, \beta]$.

Proposition 5.14. *The groups $\mathcal{D}_w$ are branch over the groups L_w, i.e., L_w contains the geometric direct product $L_{s(w)} \times L_{s(w)}$ and is of finite index in $\mathcal{D}_w$.*

Proof. The commutator $[\gamma, \beta]$ is equal to $\left((\beta\alpha)^2, 1\right)$ if the first letter of w is 0, or to $\left(1, (\beta\gamma)^2\right)$, otherwise.

We have $[\gamma, \beta^\alpha] = \gamma\alpha\beta\alpha\gamma\alpha\beta\alpha = [\gamma, \beta] \cdot [\beta, \alpha]^{\gamma\beta} \cdot [\beta, \alpha] \in L_w$. The element $[\gamma, \beta^\alpha]$ is equal to $\left((\beta\gamma)^2, 1\right)$, if the first letter of w is 0 and to $\left(1, (\beta\alpha)^2\right)$, otherwise.

Taking conjugates of these elements by α, we see that

$$\left\{\left((\beta\alpha)^2, 1\right), \left(1, (\beta\alpha)^2\right), \left((\beta\gamma)^2, 1\right), \left(1, (\beta\gamma)^2\right)\right\} \subset L_w.$$

The projection of the first level stabilizer of $\mathcal{D}_w$ is equal to $\mathcal{D}_{s(w)}$, hence we get that L_w contains the geometric direct product $L_{s(w)} \times L_{s(w)}$.

The group $\mathcal{D}_w/L_w$ is a quotient of

$$\left\langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta)^2 = (\gamma\beta)^2 = (\alpha\gamma)^4 = 1 \right\rangle,$$

which is isomorphic to $\langle \alpha, \gamma \rangle \times \langle \beta \rangle \cong D_4 \times C_2$, hence is finite of order at most 16. □

Proposition 5.15. *The groups $\mathcal{D}_w$ are just-infinite, i.e., every non-trivial subgroup of $\mathcal{D}_w$ is of finite index. Moreover, the index is always a power of 2.*

Proof. By Theorem 5.2 and Lemma 5.3 of [1] it is sufficient to show that L'_w has finite index in L_w and that index of L'_w in G_w is a power of two.

The group L_w is finitely generated, since it has finite index in $\mathcal{D}_w$. Hence, the group L_w/L'_w is a finitely generated abelian group. On the other hand, the group L_w is generated by elements of order 2 or 4 (the conjugates of $[\alpha, \beta] = (\alpha\beta)^2$ and $[\beta, \gamma] = (\beta\gamma)^2$ in $\mathcal{D}_w$). Therefore, L_w/L'_w is a torsion group, hence is finite. We see that index of L'_w in L_w is a power of 2, which implies that index of L'_w is also a power of 2, since $|\mathcal{D}_w/L_w|$ is a factor of 16. □

6. Isomorphism classes

Theorem 6.1. *The groups $\mathcal{D}_{w_1}$ and $\mathcal{D}_{w_2}$ are isomorphic if and only if they are conjugate, i.e., if and only if the sequences w_1, w_2 are cofinal. Moreover, any isomorphism $f : \mathcal{D}_{w_1} \to \mathcal{D}_{w_2}$ is induced by conjugation in* $\mathrm{Aut}(\mathsf{X}^*)$.

Proof. We have just seen that the groups $\mathcal{D}_w$ are branch. Consider now the groups $\mathcal{D}_w^{2^n}$ defined inductively by the conditions that $\mathcal{D}_w^{2^0} = \mathcal{D}_w$ and $\mathcal{D}_w^{2^n}$ is generated by the squares of $\mathcal{D}_w^{2^{n-1}}$. If $f : \mathcal{D}_{w_1} \to \mathcal{D}_{w_2}$ is an isomorphism, then $f\left(\mathcal{D}_{w_1}^{2^n}\right) = \mathcal{D}_{w_2}^{2^n}$ for all n.

Let us prove that $\mathcal{D}_w^{2^n}$ is transitive on the levels of the subtrees of the nth level. Consider the set

$$A = \{\alpha\beta\gamma, \alpha\gamma\beta, \beta\alpha\gamma, \beta\gamma\alpha, \gamma\alpha\beta, \gamma\beta\alpha\}.$$

Every element of A is level-transitive. It is also easy to see that if $g \in A$, then $g^2 = (g_0, g_1)$, where $g_0, g_1 \in A$. This proves by induction that the groups $\mathcal{D}_w^{2^n}$ are level-transitive on trees of the nth level.

This means that the groups $\mathcal{D}_w$ are *saturated*, which by Theorem 7.5 of [8] (see more explicit version in Proposition 2.10.7 of [9]) and Theorem 5.1 finishes the proof. □

Proposition 6.2. *The groups G_{w_1} and G_{w_2} are isomorphic if and only if the sequences w_1, w_2 are cofinal.*

Proof. In view of Theorem 6.1 and Proposition 5.6 it is sufficient to show that the group $G_{111\ldots}$ is not isomorphic to any group $\mathcal{D}_w$.

But the groups $\mathcal{D}_w$ are just infinite, while the group $G_{111\ldots}$ has an infinite proper quotient $\mathcal{D}_{111\ldots}$. □

Proposition 6.3. *The map $w \mapsto (\mathcal{D}_w, \alpha_w, \beta_w, \gamma_w)$ from X^ω to the space of 3-generated groups $\mathfrak{G}_3$ is injective.*

Proof. If $f : \mathcal{D}_{w_1} \to \mathcal{D}_{w_2}$ is an isomorphism such that $f(\alpha_{w_1}) = \alpha_{w_2}, f(\beta_{w_1}) = \beta_{w_2}, f(\gamma_{w_1}) = \gamma_{w_2}$, then, by Theorem 6.1, it is induced by a conjugation in $\mathrm{Aut}(\mathsf{X}^*)$, which implies $w_1 = w_2$, by Proposition 3.1. □

Theorem 6.4. *The map $w \mapsto (G_w, \alpha_w, \beta_w, \gamma_w)$ from X^ω to a subset of the space $\mathfrak{G}_3$ of 3-generated groups is a homeomorphism.*

Proof. We know from Corollary 5.11 that the map is continuous. Since the space X^ω is compact, it is sufficient to show that the map is injective.

We have to prove that if If $f : G_{w_1} \to G_{w_2}$ is an isomorphism such that $f(\alpha_{w_1}) = \alpha_{w_2}, f(\beta_{w_1}) = \beta_{w_2}, f(\gamma_{w_1}) = \gamma_{w_2}$, then $w_1 = w_2$. It is true, by Proposition 6.3 and Proposition 5.6, if both sequences w_1, w_2 have infinitely many zeros. Moreover, by Proposition 6.2, there is no such an isomorphism if one sequence has a finite number of zeros, and the other has them infinitely many. Therefore, it remains to consider the case when both sequences are cofinal to $111\ldots$. This case follows from Proposition 6.3 and the following lemma.

Lemma 6.5. *Let K_w denote the kernel of the canonical epimorphism $G_w \to \mathcal{D}_w$. Then K_w is the largest normal subgroup of G_w such that G_w/K_w is not solvable. In particular, if $f : G_{w_1} \to G_{w_2}$ is an isomorphism, then $f(K_{w_1}) = K_{w_2}$.*

Proof. Let us prove that K_w is abelian. The wreath recursion $\psi_{x_1\ldots x_n} : G_{x_1x_2\ldots} \to \mathfrak{S}(\mathsf{X}^n) \wr G_{x_{n+1}x_{n+2}\ldots}$ is injective, by definition of G_w. If $g \in G_{x_1x_2\ldots}$ belongs to $K_{x_1x_2\ldots}$, then its image in $\mathcal{D}_w$ is trivial, i.e., g acts trivially on the rooted tree, and $\psi_{x_1x\ldots x_n}(g)$ belongs to the base group $G_{x_{n+1}x_{n+2}\ldots}^{\mathsf{X}^n}$ of the wreath product. By Proposition 4.2, for any two elements $g_1, g_2 \in K_{x_1x_2\ldots}$ there exists n such that all coordinates of $\psi_{x_1\ldots x_n}(g_i) \in G_{x_{n+1}x_{n+2}\ldots}^{\mathsf{X}^n}$ belong to $\langle(\beta\gamma)^2\rangle$ (since these are the only elements of the nucleus $\mathcal{N}$, which may be trivial in $\mathcal{D}_w$). Consequently, $\psi_{x_1\ldots x_n}(g_i)$ belong to a commutative group $\langle(\beta\gamma)^2\rangle^{\mathsf{X}^n}$, hence g_1 and g_2 commute, by injectivity of $\psi_{x_1\ldots x_n}$. Hence, K_w is abelian. Actually, a closer inspection shows that K_w, for w cofinal with $111\ldots$, is isomorphic to the direct product $(\mathbb{Z}/4\mathbb{Z})^{G_w(111\ldots)}$, where the action of G_w by conjugation coincides with the action on the direct summands by the original action on the orbit of $111\ldots \in \mathsf{X}^\omega$ (compare with [2]).

Suppose now that a normal subgroup $N \lhd G_w$ contains an element not belonging to K_w. Then the group $G_w/(NK_w)$ is isomorphic to a proper quotient of $\mathcal{D}_w$, hence is solvable by Proposition 5.15. Consequently, the group G_w/N is an extension of an abelian group $NK_w/N \cong K_w/N \cap K_w$ by a solvable group G_w/NK_w, hence is solvable. □ □

7. A dynamical interpretation

7.1. **The groups $\mathcal{D}_w$ as iterated monodromy groups.** The groups $\mathcal{D}_w$ and $\mathcal{D}$ have a natural dynamical interpretation.

Let $C_0, C_1, \ldots,$ be a sequence of planes and let $\mathrm{A}_i, \mathrm{B}_i, \Gamma_i \in C_i$ be three pairwise different points in the respective plane. Let $f_i : C_i \to C_{i-1}$, for $i = 1, 2, \ldots$, be an orientation-preserving 2-fold branched covering with branching point (critical value) A_i such that

$$\begin{aligned} f_i(\mathrm{A}_i) &= \mathrm{B}_{i-1} \\ f_i(\mathrm{B}_i) &= \Gamma_{i-1} \\ f_i(\Gamma_i) &= \mathrm{B}_{i-1}. \end{aligned}$$

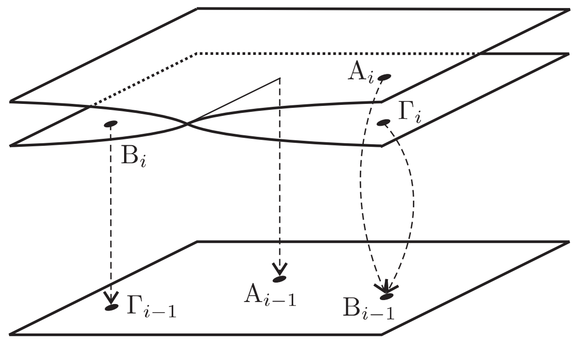

FIGURE 4. Branched coverings f_i

See Figure 4.

We denote $\mathcal{M}_i = C_i \setminus \{\mathrm{A}_i, \mathrm{B}_i, \Gamma_i\}$. Let $t \in \mathcal{M}_0$. Denote by L_n the set of preimages of the point t under the composition $f_1 \circ f_2 \circ \cdots \circ f_n : C_n \to C_0$. The union $T = \bigsqcup_{n=0}^{\infty} L_n$ has a natural structure of a rooted binary tree, where a vertex $t_n \in L_n$ is connected by an edge with the vertex $f_n(t_n) \in L_{n-1}$. The root is $t \in L_0$ and the set L_n is the nth level of the tree T.

The fundamental group $\pi_1(\mathcal{M}_0, t)$ acts naturally on each of the levels of the tree T and this action agrees with the tree structure. Namely, the image of a point $z \in L_n$ under the action of a loop $\gamma \in \pi_1(\mathcal{M}_0, t)$ is the end of the unique preimage of γ under the covering $f_1 \circ f_2 \circ \cdots \circ f_n$ beginning at z. The obtained automorphism group of the tree T is called the *iterated monodromy group* of the sequence $f_1, f_2, \ldots$, and is denoted $\mathrm{IMG}\,(f_1, f_2, \ldots)$.

Let α be a small simple loop going in positive direction around the point A_0 connected to the basepoint t by a path. Similarly, we define the elements β and γ of $\pi_1(\mathcal{M}_0, t)$ as small simple positive loops around B_0 and Γ_0, respectively.

It is easy to see, just looking at the tree of preimages of the points A_0, B_0 and Γ_0 and at the branching degrees, that the respective elements α, β and γ of the iterated monodromy group are conjugate to the elements g_0, g_1, g_2, respectively (see their definition in Section 3).

Consequently, $\mathrm{IMG}\,(f_1, f_2, \ldots)$ is isomorphic to $\mathcal{D}_w$ for some $w \in \{0,1\}^\omega$, by Proposition 3.1.

Let us describe how one can find an appropriate sequence $w = x_1 x_2 \ldots$ (which is obviously not unique).

Let $l_{\mathrm{A}_0}, l_{\mathrm{B}_0}$ and l_{Γ_0} be simple disjoint paths in $\mathcal{M}_0$ connecting infinity with the points A_0, B_0 and Γ_0, respectively.

The f_1-preimage of l_{A_0} is a path passing through the critical point of f_1 and dividing the plane C_1 into two pieces. The two preimages A_1 and Γ_1 of the point B_0 belong two different pieces. Let us denote the piece containing A_1 by S_0. The other piece, containing Γ_0, will be denoted S_1.

We have then two possibilities: either B_1 belongs to S_0, or it belongs to S_1. In the first case we put $x_1 = 0$, in the second $x_1 = 1$.

The preimage $f_1^{-1}(l_{\mathrm{B}_0})$ is a disjoint union of two paths, connecting the preimages A_0 and Γ_0 of B_0 to infinity. We will denote these preimages by l_{A_1} and l_{Γ_1}, respectively. Similarly, l_{B_1} is the preimage of l_{Γ_0}, connecting B_1 to infinity.

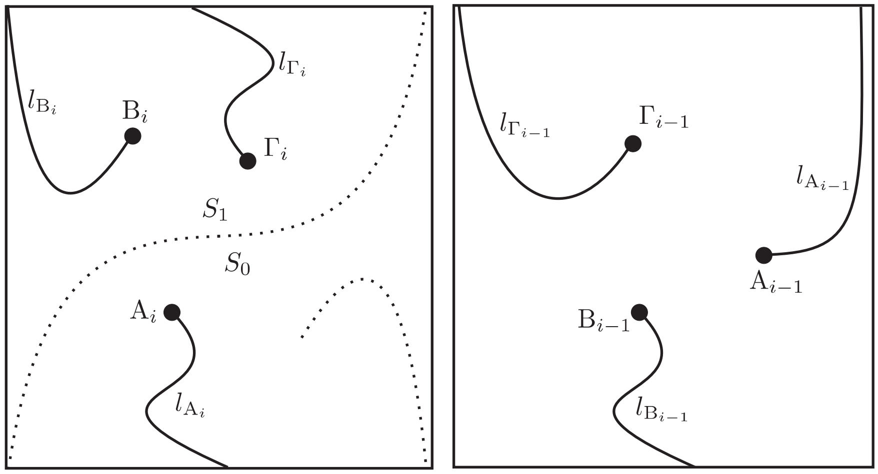

FIGURE 5. Spiders

We get in this way a collection $\{l_{A_1}, l_{B_1}, l_{\Gamma_1}\}$ (a *spider*) of the paths connecting infinity with the points A_1, B_1 and Γ_1 in the plane C_1. We can use these paths, in the same way as we used the paths $\{l_{A_0}, l_{B_0}, l_{\Gamma_0}\}$, to construct the next spider $\{l_{A_2}, l_{B_2}, l_{\Gamma_2}\}$, to partition the plane C_2 into two parts S_0, S_1, and to get the next letter x_2 of the word w.

See Figure 5 for a picture of these curves.

Proposition 7.1. *Let $\{l_{A_i}, l_{B_i}, l_{\Gamma_i}\}$ be the spiders, constructed above and let $w = x_1x_2\ldots$ be the obtained sequence. Then the iterated monodromy group* $\mathrm{IMG}\,(f_1, f_2, \ldots)$ *is isomorphic to $\mathcal{D}_w$.*

Proof. For every $t_n \in L_n$, let $\Lambda(t_n) = a_1a_2\ldots a_n$ be the *itinerary* of t_n, i.e., such a sequence that $a_i = 0$ if $f_i \circ \cdots \circ f_n(t_n)$ belongs to the sector S_0 of the plane C_i and $a_i = 1$ if it belongs to S_1. We set $\Lambda(t) = \emptyset$.

It follows directly from the definition that if $\Lambda(t_n) = a_1a_2\ldots a_n$, then $\Lambda(f_n(t_n)) = a_1a_2\ldots a_{n-1}$. This easily implies, that the map $\Lambda : T \to \{0,1\}^*$ is an isomorphism of the rooted trees.

Let α, β, γ be small simple loops around the points A_0, B_0, Γ_0 connected to the basepoint t_0 by paths disjoint with the spider $(l_{A_i} \cup l_{B_i} \cup l_{\Gamma_i})$.

An easy inductive argument shows then that the isomorphism $\Lambda : T \to \{0,1\}^*$ conjugates the action of α, β, γ on T with α_w, β_w and γ_w, respectively. □

It is also easy to deduce from the proof of the proposition that every group $\mathcal{D}_w$ can be realized as the iterated monodromy group of a sequence $f_1, f_2, \ldots$.

7.2. **An index 2 subgroup of $\widetilde{\mathcal{D}}$ as an iterated monodromy group.** Let us choose any complex structure on the plane C_0, identifying it with $\mathbb{C}$. The complex structure on C_i is defined then as the pull-back of the complex structure on C_0 by the map $f_1 \circ f_2 \circ \cdots f_i$. Then the maps f_i become quadratic polynomials. Applying the respective affine transformations, we may assume that A_i coincides with 0 and B_i coincides with $1 \in \mathbb{C}$. Let us denote by p_i the complex number, identified then with Γ_i. Then f_i is a quadratic polynomial such that

(1) its critical value is 0,

(2) $f_i(0) = 1$,
(3) $f_i(1) = p_{i-1}$,
(4) and $f_i(p_i) = 1$.

The first two conditions imply that f_i is a quadratic polynomial of the form $(az+1)^2$. The last condition implies that $ap_i + 1 = -1$ (since $p_i \neq 0$), hence $a = -2/p_i$ and $p_{i-1} = \left(1 - \frac{2}{p_i}\right)^2$.

We conclude also that $f_i(z) = \left(1 - \frac{2z}{p_i}\right)^2$.

We get thus a map

$$F : \begin{pmatrix} z \\ p \end{pmatrix} \mapsto \begin{pmatrix} \left(1 - \frac{2z}{p}\right)^2 \\ \left(1 - \frac{2}{p}\right)^2 \end{pmatrix}.$$

The polynomials $f_n(z)$ are then the first coordinates of iterations of F.

The map F can be extended to a map from $\mathbb{CP}^2$ to itself, which is defined in homogeneous coordinates by the formula

$$F : [z : p : u] \mapsto \left[(p-2z)^2 : (p-2u)^2 : p^2\right]. \tag{3}$$

The Jacobian of the map F is

$$\begin{vmatrix} -4(p-2z) & 0 & 0 \\ 2(p-2z) & 2(p-2u) & 2p \\ 0 & -4(p-2u) & 0 \end{vmatrix} = -32(p-2z)(p-2u)p,$$

hence the critical set is the union of the three lines $p = 2z, p = 2u$ and $p = 0$.

The orbits of these lines are

$$\{p = 2z\} \mapsto \{z = 0\} \mapsto \{z = u\} \mapsto \{z = p\} \mapsto \{z = u\},$$
$$\{p = 2u\} \mapsto \{p = 0\} \mapsto \{u = 0\} \mapsto \{p = u\} \mapsto \{p = u\}.$$

Hence, the post-critical set of the map F is the union of the lines $z = 0, p = 0, u = 0, p = z, p = u, z = u$. (Or, in non-homogeneous coordinates, the lines $z = 0, z = 1, p = 0, p = 1, p = z$ and the line at infinity.) Thus, it is post-critically finite (a rational mapping of $\mathbb{CP}^n$ is called post-critically finite, if its post-critical set is an algebraic variety). Note that not so many examples of post-critically finite holomorphic maps $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ are known (see [**?**]).

Let us denote by $\widehat{\mathcal{D}}$ the index 2 subgroup of $\widetilde{\mathcal{D}}$ generated by $\{ab, ac\} \cup \mathcal{D}$. We have the following interpretation of $\widehat{\mathcal{D}}$. Note that $\widehat{\mathcal{D}}/\mathcal{D} < H$ is isomorphic to the group generated by A and B, which was considered in the proof of Proposition 4.7.

Theorem 7.2. *The group* $\mathrm{IMG}\,(F)$ *is isomorphic to* $\widehat{\mathcal{D}}$. *The group* $\mathrm{IMG}\left(\left(1 - \frac{2}{p}\right)^2\right)$ *is isomorphic to* $\widehat{\mathcal{D}}/\mathcal{D}$.

7.3. **Limit space of $\mathcal{D}$.** We have seen that the groups $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ (and thus $\widehat{\mathcal{D}}$) are contracting. One can therefore define their limit spaces $\mathcal{J}_{\mathcal{D}}$, $\mathcal{J}_{\widetilde{\mathcal{D}}}$ and $\mathcal{J}_{\widehat{\mathcal{D}}}$.

One can show that the limit space of $\mathcal{J}_{\widehat{\mathcal{D}}}$ is homeomorphic to the Julia set of the rational function $F : \mathbb{CP}^2 \mapsto \mathbb{CP}^2$, considered above.

Consider the projection $P : \mathbb{CP}^2 \setminus \{[1:0:0]\} \mapsto \mathbb{CP}^1$ given by

$$P : [z : p : u] \mapsto [p : u].$$

It follows from the definition (3) of F that the projection semiconjugates F with the rational function $p \mapsto \left(1 - \frac{2}{p}\right)^2$ on $\mathbb{CP}^1$.

This semi-direct structure of F is reflected in Theorem 7.2 by relation $\mathrm{IMG}\,(F)\,/\mathcal{D} \cong \mathrm{IMG}\left(\left(1 - \frac{2}{p}\right)^2\right)$.

It also follows that the map P projects the Julia set $\mathcal{J}_F$ of F onto the Julia set of $(1 - 2/p)^2$, which coincides with the whole sphere $\mathbb{CP}^1$.

The limit space of $\mathcal{J}_{\mathcal{D}}$ is not connected, since the group $\mathcal{D}$ is not level-transitive. The connected components of $\mathcal{J}_{\mathcal{D}}$ are homeomorphic to the fibers of the projection $\mathcal{J}_F \to \mathcal{J}_{(1-2/p)^2}$. Figure 6 shows some of the fibers. The subscripts show the image of the fiber under the projection map onto $\mathbb{CP}^1$.

Note that the fiber above $2i$ is the Julia set of the polynomial $(1 - z/i)^2$, which is affine-conjugate with the polynomial $z^2 + i$. The fiber above 1 is the limit space of the Grigorchuk group $\mathcal{D}_{11\ldots}$. It coincides as a topological space with the Julia set of $(1 - 2z)^2$, which is conjugate with the polynomial $z^2 - 2$. However, it is different as an orbispace. (For orbispace structure on the limit spaces, see [9].)

There is an interesting relation of the function F and groups $\mathcal{D}_w$ and $\widehat{\mathcal{D}}$ with with Thurston's theorem on topological polynomials. More on this see the paper [2].

More about the limit spaces of the groups $\mathcal{D}, \widehat{\mathcal{D}}$ and $\widetilde{\mathcal{D}}$ will be written in a separate publication.

8. Another family

If we start from a different post-critical dynamics, we get another family of groups with similar properties. We will only state the properties of this family. The proofs are very similar to the proofs for the family $\{\mathcal{D}_w\}$. More details can be found in the preprint [10].

Let us take, for instance the dynamics of the "Douady rabbit". Let $f_i : C_i \to C_{i-1}$ be orientation-preserving 2-fold branched coverings with critical points $\Gamma_i \in C_i$ such that

$$\begin{aligned} f_i(\Gamma_i) &= \mathrm{A}_{i-1} \\ f_i(\mathrm{A}_i) &= \mathrm{B}_{i-1} \\ f_i(\mathrm{B}_i) &= \Gamma_{i-1}, \end{aligned}$$

where $\mathrm{A}_i, \mathrm{B}_i, \Gamma_i \in C_i$ are pairwise distinct points.

Small loops α, β, γ around $\mathrm{A}_0, \mathrm{B}_0, \Gamma_0$, respectively, act on the tree of preimages of the basepoint by automorphisms conjugate to the automorphisms

$$g_0 = \sigma(1, g_2), \quad g_1 = (1, g_0), \quad g_2 = (1, g_1).$$

An analog of Proposition 3.1 is true. Any such triple α, β, γ is simultaneously conjugate with exactly one triple $\alpha_w, \beta_w, \gamma_w$, for $w \in \mathsf{X}^\omega$, defined by the recursion

$$\begin{aligned} \alpha_w &= \sigma\left(1, \gamma_{s(w)}\right), \\ \beta_w &= \begin{cases} \left(1, \alpha_{s(w)}\right) & \text{if the first letter of } w \text{ is } 1, \\ \left(\alpha_{s(w)}, 1\right) & \text{if the first letter of } w \text{ is } 0, \end{cases} \\ \gamma_w &= \left(1, \beta_{s(w)}\right). \end{aligned}$$

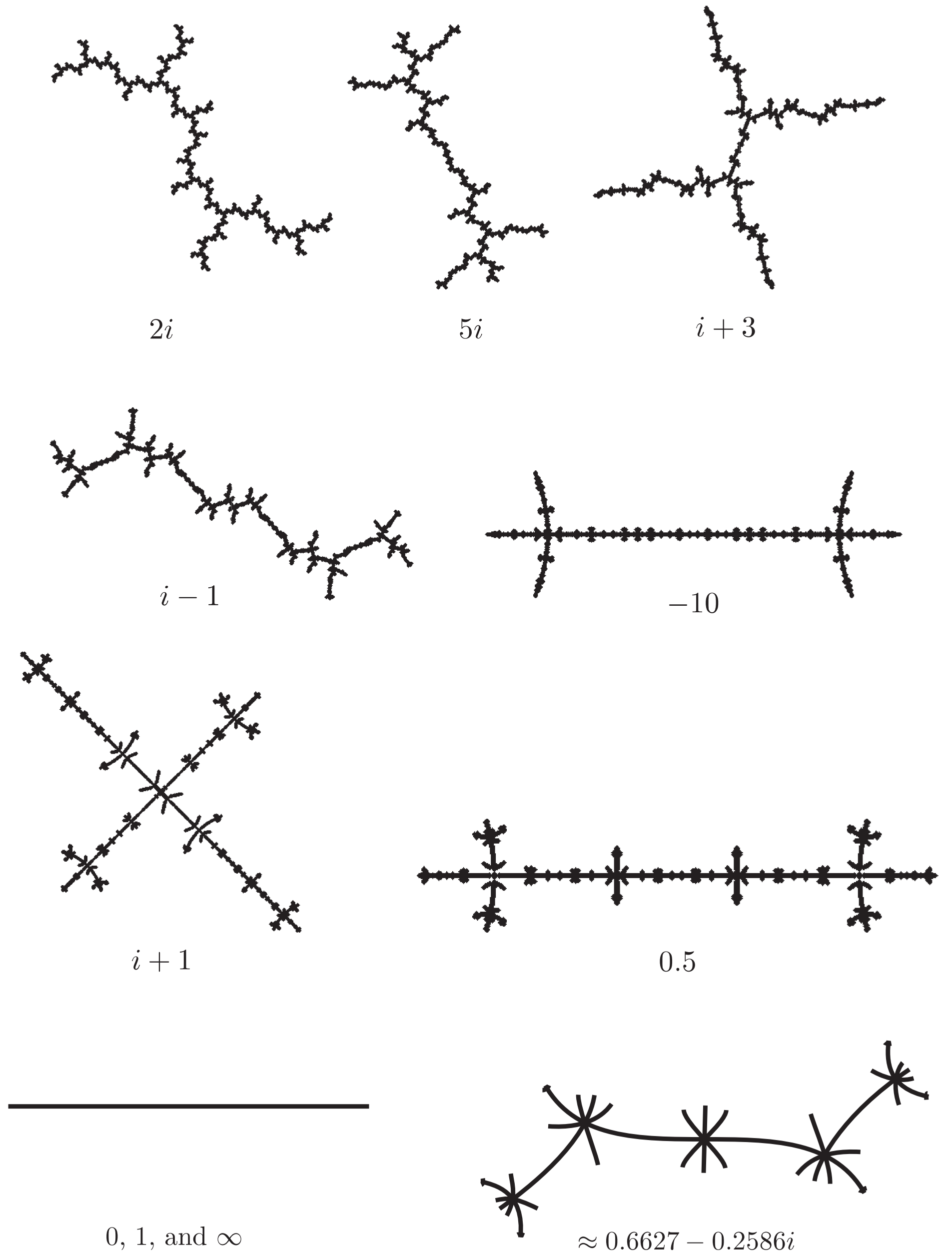

Figure 6. Connected components of $\mathcal{J}_{\mathcal{D}}$

Let us denote by $\mathcal{R}_w$ the group generated by $\alpha_w, \beta_w, \gamma_w$. Again, as for the family $\mathcal{D}_w$, we get that every group $\mathcal{R}_{w_0}$ is conjugate to at most countable number of groups $\mathcal{R}_w$.

The groups $\mathcal{R}_w$ are *weakly branch* over their commutator subgroup, i.e., $\mathcal{R}'_w$ geometrically contains $\mathcal{R}'_{s(w)} \times \mathcal{R}'_{s(w)}$. This can be used, in the same way as in Theorem 6.1, to prove that two groups $\mathcal{R}_{w_1}$ and $\mathcal{R}_{w_2}$ are isomorphic if and only if they are conjugate in $\mathrm{Aut}(\mathsf{X}^*)$. In particular, this implies that the isomorphism classes of the family $\{\mathcal{R}_w\}$ are at most countable.

If we repeat the computations of Subsection 7.2 for our situation, we get the two-dimensional map

$$\left(\begin{array}{c} z \\ p \end{array}\right) \mapsto \left(\begin{array}{c} 1-\frac{z^2}{p^2} \\ 1-\frac{1}{p^2} \end{array}\right),$$

whose iterated monodromy group $\widehat{\mathcal{R}}$ is generated by the transformations

$$\begin{aligned} \alpha &= \sigma(1,\gamma,1,\gamma) \\ \beta &= (\alpha,1,1,\alpha) \\ \gamma &= (1,\beta,1,\beta), \\ S &= (1,1,T,T) \\ T &= \pi\left(T^{-1}S^{-1}\gamma\beta, T^{-1}S^{-1}\gamma\beta\gamma^{-1}\alpha\gamma, \alpha, 1\right). \end{aligned}$$

The subgroup generated by $\alpha,\beta,\gamma \in \widehat{\mathcal{R}}$ is the universal group $\mathcal{R}$ of the family $\{\mathcal{R}_w\}$, i.e., Proposition 4.1 holds for it.

The group $\widehat{\mathcal{R}}$ is level-transitive and $\mathcal{R}$ is its normal subgroup. This, in the same way as in Proposition 4.6, implies that the two groups $\mathcal{R}_{w_1}$ and $\mathcal{R}_{w_2}$ are conjugate if the sequences w_1, w_2 belong to one orbit of the group $\widehat{\mathcal{R}}/\mathcal{R}$, which is defined by the recursion

$$s = (1,t), \qquad t = \sigma(t^{-1}s^{-1},1).$$

This group is isomorphic to $\mathrm{IMG}\left(1-1/p^2\right)$.

We get in this way that for any $w_1 \in \mathsf{X}^\omega$ the set of sequences $w \in \mathsf{X}^\omega$ such that $\mathcal{R}_{w_1}$ and $\mathcal{R}_{w_2}$ are isomorphic is dense in X^ω.

The group $\widehat{\mathcal{R}}$ does not describe the isomorphism classes of the family $\{\mathcal{R}_w\}$ completely. One can construct a bigger group $\widetilde{\mathcal{R}}$, for which the analog of Proposition 4.6 holds. It is the group generated by $\mathcal{R}$ and the automorphisms

$$\begin{array}{ll} a = \pi(c,c,1,1), & I_0 = \left(I_2c\gamma^{-1}, I_2c, I_2\gamma^{-1}, I_2\right) \\ b = (1,1,a,a), & I_1 = (I_0,I_0,I_0,I_0) \\ c = \left(1,\beta,b\beta^{-1},b\right), & I_2 = (I_1,I_1,I_1,I_1). \end{array}$$

We have the following relations

$$\begin{array}{lll} \alpha^a = \alpha, & \beta^a = \beta^\alpha, & \gamma^a = \gamma, \\ \alpha^b = \alpha, & \beta^b = \beta, & \gamma^b = \gamma^\beta, \\ \alpha^c = \alpha^\gamma, & \beta^c = \beta, & \gamma^c = \gamma. \end{array}$$

and

$$\begin{array}{lll} \alpha^{I_0} = \alpha^{-1}, & \beta^{I_0} = \beta, & \gamma^{I_0} = \gamma \\ \alpha^{I_1} = \alpha, & \beta^{I_1} = \beta^{-1}, & \gamma^{I_1} = \gamma \\ \alpha^{I_2} = \alpha, & \beta^{I_2} = \beta, & \gamma^{I_2} = \gamma^{-1}. \end{array}$$

The group $\widehat{\mathcal{R}}$ is a subgroup of $\widetilde{\mathcal{R}}$, since $T = \beta b^{-1}a$ and $S = \gamma c^{-1}b$. One can also prove that the group $\widetilde{\mathcal{R}}$ is contracting.

We get in this way the following (possibly partial) description of the isomorphism classes of the family $\{\mathcal{R}_w\}$.

Theorem 8.1. *Let H be the group generated by the automorphisms*

$$a = \sigma(c,1), \quad b = (1,a), \quad c = (1,b)$$

and

$$r_0 = (r_2 c, r_2), \quad r_1 = (r_0, r_0), \quad r_2 = (r_1, r_1).$$

If sequences $w_1, w_2 \in \mathsf{X}^\omega$ *belong to one* H*-orbit, then the groups* $\mathcal{R}_{w_1}$ *and* $\mathcal{R}_{w_2}$ *are isomorphic.*

I do not know if this is a complete description of the isomorphism classes.

The situation with the relations in the groups $\mathcal{R}_w$ is even better than for the family $\{\mathcal{D}_w\}$.

Theorem 8.2. *Let*

$$R_0 = \left\{ \left[\beta^{\alpha^{2n}}, \gamma\right], \left[\beta^{\alpha^{2n+1}}, \beta\right], \left[\gamma^{\alpha^{2n+1}}, \gamma\right] \ : \ n \in \mathbb{Z} \right\}$$

and

$$R_1 = \left\{ \left[\beta^{\alpha^{2n+1}}, \gamma\right], \left[\beta^{\alpha^{2n+1}}, \beta\right], \left[\gamma^{\alpha^{2n+1}}, \gamma\right] \ : \ n \in \mathbb{Z} \right\},$$

and let φ_i *be the endomorphisms of the free group*

$$\begin{array}{rclcrcl} \varphi_0(\alpha) & = & \alpha\beta\alpha^{-1}, & \quad & \varphi_1(\alpha) & = & \beta, \\ \varphi_0(\beta) & = & \gamma, & & \varphi_1(\beta) & = & \gamma, \\ \varphi_0(\gamma) & = & \alpha^2, & & \varphi_1(\gamma) & = & \alpha^2. \end{array}$$

Then for every sequence $w = x_1 x_2 \ldots \in \mathsf{X}^\omega$ *the set*

$$\bigcup_{n=1}^{\infty} \varphi_{x_1} \circ \varphi_{x_2} \circ \cdots \circ \varphi_{x_{n-1}}(R_{x_n})$$

is a set of defining relations of the group $\mathcal{R}_w$*.*

Minimality of the set $\{\mathcal{R}_w\}$ is also true (without any need to replace "bad groups" like in the family $\{\mathcal{D}_w\}$).

Proposition 8.3. *Let* $w_1, w_2 \in \{0,1\}^\omega$*. For any finite set of relations and inequalities between the generators* $\alpha_{w_1}, \beta_{w_1}$ *and* γ_{w_1} *there exist generators* h_0, h_1 *and* h_2 *of the group* $\mathcal{R}_{w_2}$ *such that the same set of relations and inequalities hold for the generators* h_0, h_1 *and* h_2 *in the group* $\mathcal{R}_{w_2}$*.*

The limit spaces of $\widehat{\mathcal{R}}$ and $\mathcal{R}$ can be also described in a way similar to the description of $\mathcal{J}_{\widehat{\mathcal{D}}}$ and $\mathcal{J}_{\mathcal{D}}$. The limit space $\mathcal{J}_{\widehat{\mathcal{R}}}$ is homeomorphic to the Julia set of the function

$$F(z,p) = \left(1 - \frac{z^2}{p^2}, 1 - \frac{1}{p^2}\right),$$

which is projected by $P : (z,p) \mapsto p$ onto the Julia set of the rational function $1 - 1/p^2$. The fibers of the projection are homeomorphic to the connected components of the limit space $\mathcal{J}_{\mathcal{R}}$. See Figure 7 for the Julia set of $1 - \frac{1}{p^2}$ and the fibers of the Julia set of F over the respective points of the Julia set of $1 - 1/p^2$.

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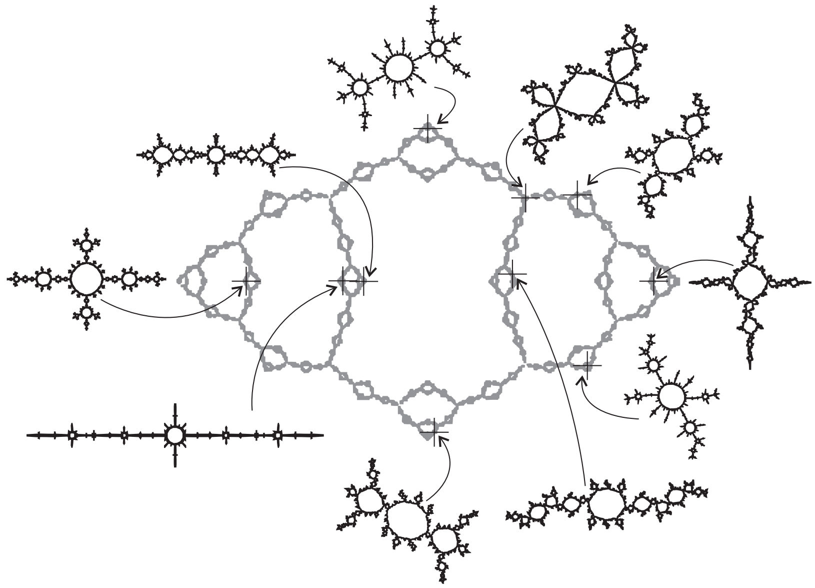

FIGURE 7. Limit space of $\widehat{\mathcal{R}}$

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