Note Title

Banach Spaces

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Definition

We refer to a complete normed vector space as a Banach space.

Definitions

(i) Given a sequence (x_n) in a vector space X, we refer to the sum $\sum_{n=1}^{\infty} x_n$ as a series.

(ii) If X is normed, we say the series $\sum_{n=1}^{\infty} x_n$ converges to $x \in X$ if the partial sums

$$S_N = \sum_{n=1}^{N} x_n$$

converge to x in X; i.e., if

$$\lim_{N \to \infty} \| S_N - x \|_X = 0.$$

We say that $\sum_{n=1}^{\infty} x_n$ is <u>summable</u> to x, and write

$$\sum_{n=1}^{\infty} x_n = x.$$

We see directly from the definition that each series can be expressed as a sequence (of partial sums), and correspondingly the limit of a sequence

$$\lim_{n\to\infty} x_n = x$$

can be expressed as a series:

$$\sum_{n=2}^{\infty} (x_n - x_{n-1}) = (\cancel{x_2} - x_1) + (\cancel{x_3} - \cancel{x_2}) + (x_4 - \cancel{x_3}) + \cdots$$

$$= \lim_{n\to\infty} x_n - x_1 = x - x_1$$

$$\Rightarrow x = x_1 + \sum_{n=2}^{\infty} (x_n - x_{n-1}).$$

Finally, we say that a series $\sum_{n=1}^{\infty} x_n$ is <u>absolutely summable</u> in X if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.