Newton's Method

Note Title 7/29/2015

The goal of this method is to solve the algebraic equation

$$f(x) = 0$$

for some root x, and we assume we have a reasonably good initial guess (approximation), which we'll denote x_1.

If f is differentiable at x_1, then

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + o(|x - x_1|),$$

where the "little-o" notation

$$h(z) = o(|z|)$$

means

$$\lim_{z \to 0} \frac{h(z)}{|z|} = 0$$

So h goes to 0 faster than z. For our equation $o(|x - x_1|)$ is smaller, for x near x_1, than $|x - x_1|$.

This means our equation is approximately

$$\underbrace{f(x)}_{0} \cong f(x_1) + f'(x_1)(x - x_1)$$

(since x is a root of f)

We have

$$0 \cong f(x_1) + f'(x_1)(x - x_1)$$

Solving for x: $x \cong x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

This suggests we might be able to improve our initial guess by making a second guess (approximation)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Likewise, we can make third, fourth etc. approximations, giving us a sequence (x_n) obtained by iteration: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

This iterative method is called Newton's method.

Likewise, if we have a system of k equations for k variables

$$f(x) = 0, \qquad x, f \in \mathbb{R}^k$$

we have (by the same argument)

$$x_{n+1} = x_n - \left(Df(x_n)\right)^{-1} f(x_n),$$

where $Df(x)$ denotes the Jacobian matrix

$$
Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots & & & \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{pmatrix}.
$$

In either case (for a single equation or a system) we get a recursion formula of the form $x_{n+1} = T(x_n)$ for an appropriate map $T(x)$. Notice that if x is the exact solution we get

$$x = T(x).$$

Since T maps x to itself we say x is a *fixed point* of T.