Note Title 7/31/2015

# Contraction Mapping Theorem

## Definition

We say a map $T: M \to M$ for some metric space $(M, d)$ is a contraction (sometimes referred to as a strict contraction) if there is some constant $0 \le \alpha < 1$ so that

$$d(T(x), T(y)) \le \alpha \, d(x, y)$$

for all $x, y \in M$.

## Example 1

If $X$ is any normed vector space the map

$$T(x) = \frac{1}{2}x$$

is a contraction. To see this, we compute

$$d(T(x), T(y)) = \|Tx - Ty\| = \|\tfrac{1}{2}x - \tfrac{1}{2}y\|$$

$$= \frac{1}{2}\|x - y\| = \frac{1}{2}d(x,y),$$

which means $d(T(x), T(y)) \le \alpha\, d(x,y)$ for $\alpha = 1/2$.

## Example 2

Take $M = (-1, 1)$ with the usual (absolute value) metric. The map

$$T(x) = \frac{1}{2}x + \frac{1}{8}x^2$$

is a contraction. To see this, notice that

$$d(T(x), T(y)) = \left| \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{2}y - \frac{1}{8}y^2 \right|$$

$$= \left| \frac{1}{2}(x-y) + \frac{1}{8}(x^2 - y^2) \right| \leq \frac{1}{2}|x-y| + \frac{1}{8}\underbrace{|x+y|}_{\leq 2}|x-y|$$

$$\leq \frac{1}{2}|x-y| + \frac{1}{4}|x-y| = \frac{3}{4}|x-y| = \frac{3}{4}\rho(x,y).$$

In stating the Contraction Mapping Theorem we'll use the notation $T^n$ to denote composition. So $T^2 = T \circ T$, which means $T^2(x) = T(T(x))$. More generally,

$$T^n = \underbrace{T \circ T \circ \dots \circ T}_{T \text{ appears } n \text{ times}}$$

The sequence of iterates $(T^n(x))$ is called the <u>orbit</u> of $x$ under $T$.

<u>Theorem 7.13</u> (Contraction Mapping Theorem)

Let $(M,d)$ be a complete metric space, and let $T: M \to M$ be a (strict) contraction. Then $T$ has a unique fixed point $x$. Moreover, given any point $x_0 \in M$

$$\lim_{n\to\infty} T^n(x_0) = x.$$

(I.e., $x$ can be obtained by iteration of $x_{n+1} = T(x_n)$.)