Note Title 8/2/2015

Application to ODEs, I

We consider ODE of the form

$$\frac{dy}{dx} = f(x, y) ; \qquad y(0) = y_0$$

where $x \in \mathbb{R}$, $y, f \in \mathbb{R}^n$.

We've shown that this equation can be expressed as

$$y(x) = y_0 + \int_0^x f(t, y(t))\, dt .$$

In this case, our iteration is

$$y_{n+1}(x) = y_0 + \int_0^x f(t, y_n(t))\,dt,$$

and we see that this is $y_{n+1}(x) = T(y_n(x))$ with

$$T(y) = y_0 + \int_0^x f(t, y(t))\,dt.$$

If we can show that T is a contraction on an appropriate metric space, we'll be able to conclude that our ODE has a unique solution.

To see that this will require some assumptions on f, consider the equation

$$\frac{dy}{dx} = y^{2/3} \quad ; \qquad y(0) = 0 .$$

I.e., $f(x,y) = y^{2/3}$. We can solve this exactly by separation:

$$\int y^{-2/3}\,dy = \int dx$$

$$3y^{1/3} = x + C$$

$$y(0) = 0 \Rightarrow 0 = C \Rightarrow$$

$$3y^{1/3} = x \Rightarrow y(x) = \frac{1}{27}x^3.$$

But clearly $y(x) \equiv 0$ is also a solution.
So we don't have uniqueness, which means the Contraction Mapping Theorem must not apply.

In fact, we can construct an infinite family of solutions: for any $K \geq 0$

$$y(x) = \begin{cases} 0 & 0 \leq x < K \\ \frac{1}{27}(x-K)^3 & x \geq K \end{cases}$$