Note Title 8/7/2015

Topological Characterization of Compactness, II

Theorem 8.9

A metric space (M,d) is compact if and only if it satisfies either (a) or (b) (and hence both) from Lemma 8.8.

Proof

First, we did the case $(\Leftarrow)$ in the previous lecture.

For ($\Rightarrow$) we assume M is compact, and we need to show either (a) or (b). We'll focus on (a). Let $\mathcal{G}$ denote an open cover of M, and (in order to get a contradiction) assume $\mathcal{G}$ does <u>not</u> contain a finite subcover. Since M is totally bounded it can be covered by a finite collection

$$\{B_\varepsilon(x_k) : \{x_k\}_{k=1}^{n} \subset M\}$$

for any $\varepsilon > 0$. We can take $\varepsilon = 1/2$ and also use the closures of these balls, so that we have a cover by closed sets of diameter 1.

If each of these sets (a finite number total) could be covered by a finite collection of subsets of $\mathcal{G}$, then we could have a finite subcover, so there must be at least one closed set with diameter $\mathcal{L}$ that cannot be covered by a finite collection of sets in $\mathcal{G}$. Call this set A_1. Since any finite set is easily covered by a finite collection of sets, A_1 must be infinite.

Now as a subset of a totally bounded set, A_1

is totally bounded, and we can repeat the above procedure, covering A_1 by finitely many closed sets, this time with diameter $1/2$ (i.e., using balls of radius $1/4$). At least one of these cannot be covered by a finite collection of subsets of $\mathcal{G}$, and we'll call this set A_2.

Continuing in this way, we obtain a decreasing sequence of closed, totally bounded infinite sets

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

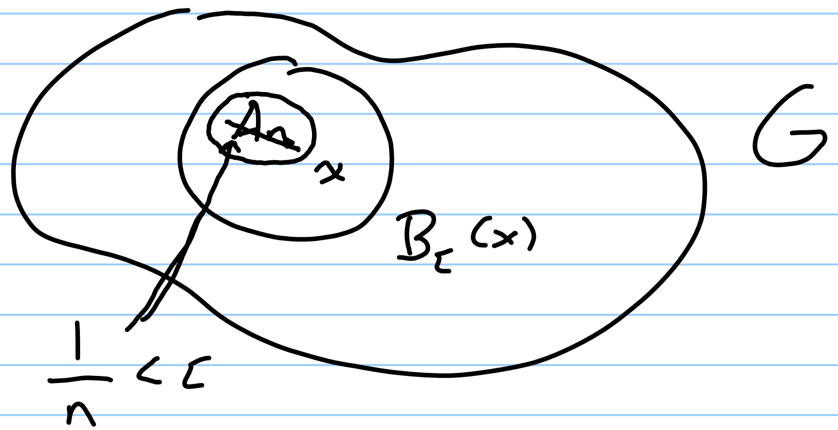
where $\operatorname{diam}(A_n) = \frac{1}{n}$. None of these sets can be covered by a finite collection of sets in $\mathcal{G}$.

We know from Theorem 7.11, which we reviewed in the previous lecture, and the fact that M is complete that

$$\bigcap_{k=1}^{\infty} A_k \neq \emptyset.$$

Take $x \in \bigcap_{k=1}^{\infty} A_k$, and notice that $x \in G \in \mathcal{G}$ for some G, because $\mathcal{G}$ is a cover of M. Also, G is open, so there exists $\varepsilon > 0$ small enough so that $B_\varepsilon(x) \subset G$. But for any n with $\frac{1}{n} < \varepsilon$ we would have

$$x \in A_n \subset B_\varepsilon(x) \subset G$$



But this means that A_n is covered by a finite collection of sets in $\mathcal{B}$ (in fact, just one, G). This is our contradiction. $\square$