Two More Characterizations of Compactness

Note Title 8/8/2015

Corollary 8.10

A metric space (M,d) is compact iff every decreasing sequence of nonempty closed sets has nonempty intersection; that is, iff whenever $F_1 \supset F_2 \supset \cdots$ is a sequence of nonempty closed sets in M, we have

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof

For ($\Longrightarrow$) if M is compact this follows from Part (b) of Lemma 8.8. For ($\Longleftarrow$) we suppose that every decreasing sequence of nonempty closed sets has nonempty intersection, and let (x_n) be any sequence in M. We need to show that (x_n) has a convergent subsequence.

Consider the sets

$$F_n = \overline{\{x_k : k \geq n\}}$$

Clearly,

$$F_1 \supset F_2 \supset \cdots$$

so $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$ by assumption. We can take $x \in \bigcap_{n=1}^{\infty} F_n$, and notice that $x \in F_n$ for all $n = 1, 2, \ldots$.

Suppose $x \in \{x_k : k \geq n\}$ $\forall n$ (i.e., not the closures). For $n=1$, we have

$$x \in \{x_k : k \geq 1\},$$

and so there exists n_1 so that $x = x_{n_1}$. But we also have $x \in \{x_k : k \geq n_1 + 1\}$, and so there exists some $n_2 > n_1$ so that $x = x_{n_2}$. Continuing this way we see that the subsequence (x_{n_j}) is simply $(x, x, x, \dots)$, which clearly converges to x.

On the other hand, if there exists n so that $x \notin \{x_k : k \geq n\}$, then $x \in \overline{\{x_k : k \geq n\}}$, and this means x can be obtained as the limit of a sequence of elements in $\{x_k : k \geq n\}$. But this will be a subsequence of (x_n). $\square$

Corollary 8.11

A metric space (M, d) is compact iff every countable open cover admits a finite subcover.

Proof

For $(\Rightarrow)$ this is true by Part (a) of Lemma 8.8 for <u>every</u> open cover, so certainly for all countable open covers. For $(\Leftarrow)$

the proof is carried out in Problem 8.72. □