# Uniform Continuity

Note Title 8/8/2015

## Definition

We say that a function $f:(M,d)\to(N,\rho)$ is uniformly continuous if given any $\varepsilon>0$ there exists $\delta>0$ so that

$$d(x,y)<\delta \implies \rho(f(x),f(y))<\varepsilon$$

$\forall\, x,y\in M$.

For a continuous function, we first must fix $x$, and $\delta$ can depend on $x$. For uniform continuity, the same $\delta$ works for all $x \in M$.

A typical example of a continuous function that is <u>not</u> uniformly continuous is $f(x) = 1/x$ on $(0,1)$. The right endpoint isn't so important, but $f(x) \to +\infty$ as $x \to 0$.

When we say that $f$ is continous in $(0,1)$ we mean that given any $x \in (0,1)$ there exists $\delta > 0$ so that

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon .$$

Notice in particular that we require $\delta < x$, because if $\delta = x$ we could take $y \to 0$ to contradict $|f(x) - f(y)| < \varepsilon$ (since $f(y)$ could be made arbitrarily large).

For this example, we see clearly how $\delta$ depends on $x$, and this is not allowed by uniform continuity.