Continuous Functions on Compact Metric Spaces

Theorem 8.15

If (M,d) is a compact metric space, then every continuous map $f:(M,d) \to (N,\rho)$ is uniformly continuous.

Proof

Let $\varepsilon > 0$, and notice that by continuity we have the following: given any $x \in M$ there exists $\delta_x > 0$ so that

$$d(x,y) < \delta_x \implies \rho(f(x), f(y)) < \varepsilon.$$

Notice that the collection

$$\left\{ B_{\frac{\delta_x}{2}}(x) : x \in M \right\}$$

is an open cover for M, and since M is compact this means there are finitely many points $\{x_i\}_{i=1}^{k} \subset M$ so that

$$M \subset \bigcup_{i=1}^{k} B_{\eta_i}(x_i), \quad \text{where} \quad \eta_i = \frac{\delta_{x_i}}{2}.$$

Set $\delta = \min\{\eta_1, \eta_2, \ldots, \eta_k\}$. Let's check that:

$$d(x,y) < \delta \implies \rho(f(x), f(y)) < 2\varepsilon$$

$\forall x, y \in M$, which means that f is uniformly continuous on M.

To see this, let $x, y \in M$ with $d(x,y) < \delta$.

We must have $x \in B_{\eta_i}(x_i)$ for some i, so

$$d(y, x_i) \le d(y, x) + d(x, x_i) < \delta + \eta_i \le 2\eta_i = \delta_{x_i}$$

(by the definition of δ). We have:

$$d(x, x_i) < \eta_i < \delta_{x_i} \implies \rho(f(x), f(x_i)) < \varepsilon$$

and

$$d(y, x_i) < \delta_{x_i} \implies \rho(f(y), f(x_i)) < \varepsilon.$$

$\implies$

$$\rho(f(x), f(y)) \le \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < 2\varepsilon,$$

and we can conclude that f is uniformly continuous on M. $\square$