Extension of Uniformly Continuous Functions

Note Title 8/9/2015

Theorem 8.16

Let D be a dense subset of a metric space (M, d), let (N, ρ) be a complete metric space, and let $f : D \to N$ be uniformly continuous. Then f extends uniquely to a uniformly continuous map $f : M \to N$, defined on all of M. Moreover, if f is an isometry, then so is the extension F.

<u>Proof</u>

Let's start with uniqueness. Suppose F and G are two such extensions but that for some $x \in M$ $F(x) \neq G(x)$. But D is dense in M, so there exists a sequence $(x_n) \subset D$ so that $x_n \longrightarrow x$ in M. Since $F = G = f$ on

D, we have

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} f(x_n)$$

and likewise

$$G(x) = \lim_{n\to\infty} G(x_n) = \lim_{n\to\infty} f(x_n),$$

so $F(x) = G(x)$.

For existence, we construct $F: M \to N$ as follows: given $x \in M$ we know by density that there is a sequence $(x_n) \subset D$ so that $x_n \to x$ in M. Since (x_n) converges it must be Cauchy in D and since f is uniformly continuous this means $(f(x_n)) \subset N$ is Cauchy in N. But N is complete, so $f(x_n) \xrightarrow{n \to \infty} y$ for some $y \in N$. We define $F(x) = y$; i.e. we carry out this process for each $x \in M$.

We need to verify that the function F defined this way is well-defined. I.e., we need to show that $F(x)$ cannot take multiple values. For this, suppose (x_n) and (z_n) are two sequences in D, both going to $x \in M$. The sequence $x_1, z_1, x_2, z_2, \ldots$ also converges to x, and so by the same considerations as above the sequence $f(x_1), f(z_1), f(x_2), \ldots$ must converge to some $w \in N$. Every subsequence of a converging series has to converge to the same value, so we must have that w is the value obtained from both (x_n) and (y_n), so $F(x)$ is well-defined.

Next, let's check that F is uniformly continuous on M. Let $\varepsilon > 0$ be given and choose $\delta > 0$ small enough so that

$$d(x', y') < \delta \implies \rho(f(x'), f(y')) < \varepsilon$$

for all $x', y' \in D$ (possible, because f is uniformly continuous on D). Also, notice that given any $x \in M$ there is $x' \in D$ so that

$$d(x, x') < \frac{\delta}{3}$$

(by density) and $\rho(F(x), f(x')) < \varepsilon$ (because $f(x_n') \to F(x)$ for some sequence $(x_n') \subset D$).

Now take any $x, y \in M$ with $d(x,y) < \frac{\delta}{3}$, and choose $x', y' \in D$ so that

$$d(x, x') < \frac{\delta}{3}$$

$$d(y, y') < \frac{\delta}{3}$$

$$\rho(F(x), f(x')) < \varepsilon$$

$$\rho(F(y), f(y')) < \varepsilon.$$

We have

$$d(x', y') \le d(x', x) + d(x, y')$$

$$\le d(x', x) + d(x, y) + d(y, y') < \delta.$$

It follows that

$$\rho(F(x), F(y)) \le \rho(F(x), f(x')) + \rho(f(x'), f(y'))$$
$$+ \rho(f(y'), F(y))$$
$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

In total, we have

$$d(x,y) < \frac{\delta}{3} \implies \rho(F(x), F(y)) < 3\varepsilon,$$

but this implies uniform continuity, because ε is arbitrary.

Finally, for the isometry take any $x, y \in M$ with associated $(x_n), (y_n) \subset D$ so that $x_n \to x$ in M and $y_n \to y$ in M.

Then

$$d(x,y) \underset{\text{Pr. 3.34}}{=} \lim_{n\to\infty} d(x_n, y_n) \underset{\substack{\text{because } f \\ \text{is an isometry} \\ \text{on } D}}{=} \lim_{n\to\infty} \rho\big(f(x_n), f(y_n)\big)$$

$$\underset{\text{Pr. 3.34}}{=} \rho\big(F(x), F(y)\big)$$

$\square$