Equivalent Metrics

Note Title 8/11/2015

In this section, we consider a single set M with multiple metrics, two of which will generally be denoted d and ρ. I.e., we'll consider the metric spaces (M, d) and (M, ρ). Since the set is the same for each space, we can map from one to the other with the identity map, which we denote i. That is,

$$i : (M, d) \to (M, \rho)$$

along with its inverse

$$i^{-1} : (M, \rho) \to (M, d)$$

defined by $i(x) = x \quad \forall\, x \in M$.

Definitions

(i) We say that d and ρ are <u>equivalent</u> if both i and i^{-1} are continuous (i.e., if i is a homeomorphism).

(ii) We say that d and ρ are <u>uniformly equivalent</u> if i and i^{-1} are both uniformly continuous (i.e., if i is a uniform homeomorphism).

(iii) We say that d and ρ are **strongly equivalent** if both i and i^{-1} are Lipschitz continuous. In this case, there exist constants c and C so that

$$c\rho(x,y) \leq d(x,y) \leq C\rho(x,y)$$

Some authors take strong equivalence as their definition of equivalence.

Since all we're doing is adding properties to i, it's clear that

strongly equivalent $\Rightarrow$ uniformly equivalent
$\Rightarrow$ equivalent

Example

Take $M = [0,1]$ and consider $d(x,y) = |x-y|$ and $\rho(x,y) = \sqrt{|x-y|}$.

For equivalence, all we need to show is that both i and i^{-1} are continuous on $[0,1]$. That is, given any $x \in [0,1]$ and any $\varepsilon > 0$ there exists $\delta_1 > 0$ so that

$$d(x,y) < \delta_1 \implies \rho(x,y) < \varepsilon$$

and $\delta_2 > 0$ so that

$$\rho(x,y) < \delta_2 \implies d(x,y) < \varepsilon.$$

In this case,

$$d(x,y) < \delta_1 \implies \rho(x,y) < \sqrt{\delta_1}$$

and likewise

$$\rho(x,y) < \delta_2 \implies d(x,y) < \delta_2^{\,2}$$

We have continuity, and so we have equivalent metrics. In fact, since δ_1 and δ_2 do not depend on x (only on ε) we have uniformly equivalent metrics.

However, we can check that d and ρ are not strongly equivalent in this case. In particular, it is not possible to find a constant C so that

$$\sqrt{|x-y|} \leq C|x-y|$$

$\forall x, y \in [0,1]$. To see this, set $t = |x-y|$ (so $t \in [0,1]$) and suppose $\sqrt{t} \leq Ct$

for all $t \in [0,1]$. This means

$$1 \leq C\sqrt{t}$$

for all $t \in (0,1]$, but this implies

$$\sqrt{t} \geq \frac{1}{C},$$

which is a contradiction, since we can take $t \to 0$.