Equivalence of Norms on Finite-Dimensional Vector Spaces

Note Title 8/13/2015

Theorem 8.22

Any two norms on a single finite-dimensional vector space V are equivalent (and so strongly equivalent).

Proof

Let V be a finite-dimensional vector space with basis $x_1, x_2, \ldots, x_n$. The approach will be to introduce a convenient norm to work with, and to show that all other norms on V are equivalent to this.

Recall that each $x \in V$ can be expressed uniquely as a linear combination of basis elements

$$x = \sum_{i=1}^{n} \alpha_i x_i ,$$

for some scalars $\{\alpha_i\}_{i=1}^{n} \subset \mathbb{R}$. In this way, we can associate each $x \in V$ with a vector of coefficients

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n.$$

Our candidate for the norm of $x \in V$ is

$$\|x\| = \sum_{i=1}^{n} |\alpha_i|.$$

Let's check that this defines a norm.

First,

$$\|x\| = 0 \iff \alpha_i = 0 \;\forall i \iff x = 0$$

(this last $\iff$ is a statement about linear independence of the basis elements $\{x_i\}_{i=1}^{n}$).

Second, for $\beta \in \mathbb{R}$

$$\|\beta x\| = \left\|\sum_{i=1}^{n} \beta \alpha_i x_i\right\| = \sum_{i=1}^{n} |\beta \alpha_i| = |\beta| \sum_{i=1}^{n} |\alpha_i|$$

$$= |\beta| \, \|x\|.$$

Last,

$$\|x+y\| = \left\| \sum_{i=1}^{n} \alpha_i x_i + \beta_i x_i \right\|$$

$$= \left\| \sum_{i=1}^{n} (\alpha_i + \beta_i) x_i \right\| = \sum_{i=1}^{n} |\alpha_i + \beta_i|$$

$$\le \sum_{i=1}^{n} \left(|\alpha_i| + |\beta_i| \right)$$

$$= \sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{n} |\beta_i| = \|x\| + \|y\|.$$

Next, consider the basis-to-basis map

$$x = \sum_{i=1}^{n} \alpha_i x_i \in V \longmapsto y = \sum_{i=1}^{n} \alpha_i e_i \in \mathbb{R}^n,$$

where the e_i are the standard Euclidean basis elements in $\mathbb{R}^n$.

With our norm, this is an isometry between $(V, \|\cdot\|)$ and $(\mathbb{R}^n, \|\cdot\|_1)$. In light of this, any sequence converging in $(\mathbb{R}^n, \|\cdot\|_1)$

corresponds with a sequence converging in $(V, \|\cdot\|)$, and consequently a set is compact in $(V, \|\cdot\|)$ iff its corresponding set is compact in $(\mathbb{R}^n, \|\cdot\|_1)$. (By the sequential characterization of compactness.) In particular, the set

$$S = \{x \in V : \|x\| = 1\}$$

is compact because it corresponds with a unit sphere in $\mathbb{R}^n$, which we know is compact by Problem 8.8.

Now, suppose $|||\cdot|||$ denotes any other norm on V. Then for $x = \sum_{i=1}^n \alpha_i x_i$ we have

$$||| \sum_{i=1}^n \alpha_i x_i ||| \leq \sum_{i=1}^n |\alpha_i| \, |||x_i|||$$

$$\leq \underbrace{\max_{1 \leq j \leq n} |||x_j|||}_{C} \sum_{i=1}^n |\alpha_i| = C \, \|x\|$$

I.e., we have

$$|||x||| \le C \|x\|$$

for this constant C.

For the other inequality, recall that for any norm we have

$$\big| |||x||| - |||y||| \big| \le |||x-y||| \le C\|x-y\|$$

(I.e., $|||\cdot|||$ is a continuous map $|||\cdot||| : (V, \|\cdot\|) \to [0,\infty)$)

In particular, $|||\cdot|||$ is continuous on S. Since S is compact, $|||\cdot|||$ must achieve its minimum on S. Since this minimum is attained it cannot be 0, because $|||x||| = 0 \iff x = 0$, and on S we are confined to $\|x\| = 1$, which implies $x \neq 0$. This means

$$|||x||| \geq c > 0 \quad \forall x \in S.$$

I.e., $\|x\| = 1 \implies |||x||| \geq c$. Now, let $x \in V$

be any element so that $x \neq 0$ and notice that

$$\left\| \frac{x}{\|x\|} \right\| = 1 \implies \left|\!\left|\!\left| \frac{x}{\|x\|} \right|\!\right|\!\right| \geq c \implies \frac{1}{\|x\|} |||x||| \geq c$$

$$\implies |||x||| \geq c\|x\|.$$

Combining these observations, we have

$$c\|x\| \leq |||x||| \leq C\|x\|,$$

which means that $|||\cdot|||$ and $\|\cdot\|$ are strongly equivalent. But since $|||\cdot|||$ was arbitrary, this means that all norms on V are strongly equivalent. $\square$