Note Title

Properties of Finite-Dimensional Vector Spaces

8/15/2015

Corollary 8.23

Let V and W be normed vector spaces with V finite dimensional. Then every linear map $T: V \to W$ is continuous.

Proof

Precisely as in the proof of Theorem 8.22

let $\{x_i\}_{i=1}^{n} \subset V$ be a basis for V, and use the norm

$$\|x\| = \sum_{i=1}^{n} |\alpha_i| \qquad \left(x = \sum_{i=1}^{n} \alpha_i x_i \right),$$

Keeping in mind that this is equivalent to any other norm on V.

For any linear map $T : (V, \|\cdot\|) \to (W, |||\cdot|||)$ we have

$$\left|\left|\left| T\left(\sum_{i=1}^{n} \alpha_i x_i\right) \right|\right|\right| = \left|\left|\left| \sum_{i=1}^{n} \alpha_i T(x_i) \right|\right|\right|$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \, \left|\left|\left| T(x_i) \right|\right|\right|$$

$$\leq \max_{1 \leq j \leq n} \left|\left|\left| T(x_j) \right|\right|\right| \underbrace{\sum_{i=1}^{n} |\alpha_i|}_{\|x\|}$$

$$\leq C \, \|x\|$$

for $C = \max_{1 \leq j \leq n} \left|\left|\left| T(x_j) \right|\right|\right|$.

We see that

$$|||Tx||| \le C\,\|x\|,$$

which means that T is bounded, and so by Theorem 8.20 is continuous. $\square$

Corollary 8.24

Any two finite-dimensional normed vector spaces of the same dimension are uniformly homeomorphic. In fact, we can even find a linear (and hence Lipschitz) homeomorphism between them.

The proof is assigned as Problem 8.87.

Corollary 8.25

Every finite-dimensional normed vector space is complete.

Proof

This follows immediately from Problem 8.86 (which will be assigned), which states that any finite-dimensional normed vector space of dimension n is linearly isometric to

$(\mathbb{R}^n, |||\cdot|||)$ for some norm $|||\cdot|||$. In this way, completeness gets inherited from completeness of $\mathbb{R}^n$. $\square$

Corollary 8.26

A finite-dimensional linear subspace of any normed vector space is always closed.

Proof

The subspace is a finite-dimensional normed vector space, and so by Corollary 8.25 is complete. But this means it is closed (by Theorem 7.9). $\square$