The Oscillation of a Function at a Point

Note Title 8/16/2015

Recall that we've already defined the oscillation of a function f on a bounded interval I as

$$\omega(f; I) := \sup_{x, y \in I} |f(x) - f(y)|$$

$$= \operatorname{diam} f(I).$$

Definition

If f is bounded in some neighborhood of a value a, we define the *oscillation of f at a* by

$$\omega_f(a) := \inf_{\substack{I \ni a \\ I \text{ open}}} \omega(f; I).$$

Notice that $\omega(f;I)$ is non-decreasing in I:

$$I \subset J \implies \omega(f;I) \leq \omega(f;J).$$

Since for any $I \ni a$ with I open we can find $h_1 > 0$ so that

$$I \subset (a-h_1, a+h_1)$$

and $h_2 > 0$ so that

$$(a-h_2, a+h_2) \subset I$$

we can express $\omega_f(a)$ as follows:

$$\omega_f(a) = \inf_{h>0} \omega(f; (a-h, a+h))$$

$$= \lim_{h \to 0^+} \omega(f; (a-h, a+h))$$

$$= \lim_{h \to 0^+} \operatorname{diam} f(B_h(a))$$

If f is unbounded in every neighborhood of a, we take $\omega_f(a) = \infty$.

Suppose f is continuous at a. Then for any $x, y \in (a-h, a+h)$ we have

$$|f(x) - f(y)| = |f(x) - f(z) + f(z) - f(y)|$$

$$\leq |f(x) - f(z)| + |f(z) - f(y)|$$

By continuity each of these goes to 0 as $h \to 0$, so $\omega_f(a) = 0$.

Likewise, if $\omega_f(a) = 0$ it means that given any $\varepsilon > 0$ we can find h small enough so that for all $x, y \in (a-h, a+h)$

$$|f(x) - f(y)| < \varepsilon.$$

In particular, we can take $y = a$ to see

$$|f(x) - f(a)| < \varepsilon,$$

and this means f is continuous at a.

In summary, f is continuous at a iff $\omega_f(a) = 0$.