A Characterization of $D(f)$

Note Title 8/16/2015

Theorem 9.2

If $f: \mathbb{R} \to \mathbb{R}$, then $D(f)$ is the countable union of some collection of closed sets in $\mathbb{R}$. (Possibly including the empty set $\emptyset$ if f is continuous on $\mathbb{R}$.)

Proof

First, we can write $D(f)$ as a countable union:

$$D(f) = \{ a : \omega_f(a) > 0 \}$$
$$= \bigcup_{n=1}^{\infty} \{ a : \omega_f(a) \geq \tfrac{1}{n} \}$$

If we can show that sets of the form $\{ a : \omega_f(a) \geq \frac{1}{n} \}$ are closed then we're done.

To see this, we'll check that the complements,

$$\{a : \omega_f(a) < \frac{1}{n}\}$$

are open. To this end, fix n, and take any

$$x_0 \in \{a : \omega_f(a) < \frac{1}{n}\}.$$

In particular, we must have $\omega_f(x_0) < \frac{1}{n}$, so that

$$\lim_{h \to 0^+} \omega(f; (x_0 - h, x_0 + h)) < \frac{1}{n}.$$

For this to be possible, we must be able to find h small enough so that

$$\omega(f; (x_0 - h, x_0 + h)) < \frac{1}{n},$$

and this means

$$(x_0 - h, x_0 + h) \subset \left\{ a : \omega_f(a) < \frac{1}{n} \right\}.$$

But $(x_0 - h, x_0 + h)$ is an interval around x_0 contained in this set, so the set is open. $\square$