Note Title 8/17/2015

The Baire Category Theorem for $\mathbb{R}$

Theorem 9.3 (Baire Category Theorem for $\mathbb{R}$)

If (G_n) is a sequence of dense open sets in $\mathbb{R}$, then $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. In fact, $\bigcap_{n=1}^{\infty} G_n$ is dense in $\mathbb{R}$.

Proof

Let $x_0 \in \mathbb{R}$, and take I_0 to be any open interval containing I_0. We'll show

$$I_0 \cap \left(\bigcap_{n=1}^{\infty} G_n \right) \neq \emptyset .$$

Since I_0 can be arbitrarily small, this will mean that $\bigcap_{n=1}^{\infty} G_n$ is dense in $\mathbb{R}$ (which of course implies that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$).

Starting with G_1, since it's dense we know $I_0 \cap G_1 \neq \emptyset$. This means there must be some

$x_1 \in I_0 \cap G_1$, and since $I_0 \cap G_1$ is open there must be some interval $I_1 \ni x_1$ so that $I_1 \subset I_0 \cap G_1$. By shrinking I_1 if necessary, we can take $\operatorname{diam}(I_1) \leq 1$ and $\overline{I_1} \subset I_0 \cap G_1$.

Likewise, G_2 is dense in $\mathbb{R}$, so $I_1 \cap G_2 \neq \emptyset$, and since G_2 is open this means there is an interval $I_2 \subset I_1 \cap G_2$. By shrinking I_2 if necessary, we can take $\mathrm{diam}(I_2) \leq \frac{1}{2}$

and

$$\overline{I}_2 \subset \underline{I_1} \cap G_2 \subset \underline{I_0 \cap G_1} \cap G_2$$

Repeating this for $G_3, G_4, \ldots$ we obtain a sequence of intervals

$$\overline{I}_1 \supset \overline{I}_2 \supset \overline{I}_3 \supset \cdots$$

with $\mathrm{diam}(I_n) \leq 1/n$, and

$$\overline{I_n} \subset I_0 \cap \left(\bigcap_{k=1}^{n} G_k \right)$$

According to the Nested Interval Theorem (Theorem 7.11 (iii)) we know

$$\bigcap_{n=1}^{\infty} \overline{I_n} \neq \emptyset.$$

But if $x \in \bigcap_{n=1}^{\infty} \overline{I_n}$ then $x \in I_0 \cap \left(\bigcap_{k=1}^{\infty} G_k \right)$,

so $I_0 \cap \left(\bigcap_{k=1}^{\infty} G_k \right) \neq \emptyset$. And this is what we needed to show. $\square$