Applications of the BCT on $\mathbb{R}$, I

Application 1

We can use the BCT on $\mathbb{R}$ to give another proof that $\mathbb{R}$ is uncountable. To see this, suppose $\mathbb{R}$ is countable, in which case we can list its elements as $\{x_n\}_{n=1}^{\infty}$. The sets $G_n := \mathbb{R} \setminus \{x_n\}$ are all open and dense in $\mathbb{R}$, so we can conclude from the

BCT on $\mathbb{R}$ that $\bigcap_{n=1}^{\infty} G_n$ is dense in $\mathbb{R}$.

But since $\{x_n\}_{n=1}^{\infty}$ includes all elements of $\mathbb{R}$,

we see that

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} (\mathbb{R} \setminus \{x_n\}) = \emptyset,$$

and this is a contradiction.

Definitions

(i) We refer to a countable union of closed sets as an F_σ set.

(ii) We refer to a countable intersection of open sets as a G_δ set.

Application 2

We can use the BCT on $\mathbb{R}$ to show that any dense G_δ set on $\mathbb{R}$ is uncountable. We assume G_δ is countable, in which case we can list its elements as $G_\delta = \{x_n\}_{n=1}^{\infty}$.

At the same time, by definition, G_δ must be a countable intersection of open sets

$$G_\delta = \bigcap_{n=1}^{\infty} G_n$$

for some collection $\{G_n\}_{n=1}^{\infty}$ of open sets. But $G_\delta \subset G_n$ $\forall\, n \in \mathbb{N}$, so each G_n must be dense in $\mathbb{R}$. In this way, we see that the sets $\tilde{G}_n := G_n \setminus \{x_n\}$ will still be open and dense in $\mathbb{R}$, so by the BCT on $\mathbb{R}$ $\bigcap_{n=1}^{\infty} \tilde{G}_n$ will be dense in $\mathbb{R}$. But this leads to precisely the same contradiction as before.

because

$$\bigcap_{n=1}^{\infty} \widetilde{C_n} = \emptyset .$$