Note Title 8/18/2015

Applications of the BCT on $\mathbb{R}$, II

Corollary 9.4

$\mathbb{Q}$ cannot be written as the countable intersection of open subsets of $\mathbb{R}$. (I.e., $\mathbb{Q}$ is not a G_δ set.)

Proof

This is an immediate consequence of our

observation from the previous lecture that a dense G_δ set in $\mathbb{R}$ must be uncountable, because $\mathbb{Q}$ is countable, and so since $\mathbb{Q}$ is dense in $\mathbb{R}$ it cannot be a G_δ set. $\square$

<u>Corollary 9.5</u>

$\mathbb{R} \setminus \mathbb{Q} \neq D(f)$ for any $f: \mathbb{R} \to \mathbb{R}$.

<u>Proof</u>

We know from Theorem 9.2 that $D(f)$ is a countable union of closed sets, so if

$$\mathbb{R} \setminus \mathbb{Q} = D(f)$$

then $\mathbb{R} \setminus \mathbb{Q}$ would be a countable union of closed sets, which we can denote

$$\bigcup_{n=1}^{\infty} F_n .$$

By De Morgan's Laws, this would mean

$$\mathbb{Q} = \left(\bigcup_{n=1}^{\infty} F_n\right)^c = \bigcap_{n=1}^{\infty} F_n^c ,$$

which is a countable intersection of open sets, and this contradicts Corollary 9.4. $\square$

Corollary 9.6

If $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$, where each E_n is closed, then some E_n contains an open interval.

Proof

Each of the sets $G_n = \mathbb{R} \setminus E_n$ is open in $\mathbb{R}$, and $\bigcap_{n=1}^{\infty} G_n = \emptyset$ (because every element of $\mathbb{R}$ is taken away in the E_n's). By the BCT for $\mathbb{R}$, at least one of the G_n is not dense in $\mathbb{R}$, and we'll denote this one G_n. If $I \cap G_n \neq \emptyset$ for all open intervals $I \subset \mathbb{R}$ then G_n would be dense in $\mathbb{R}$, so there must exist some open interval $I \subset \mathbb{R}$ so that $I \cap G_n = \emptyset$. But then $I \subset E_n$

(because $E_n = G_n^c$), which is what we wanted to show. $\square$

Corollary 9.7

If $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$, then the closure of some E_n contains an interval; that is, $\operatorname{int}(\overline{E_n}) \neq \emptyset$ for some n.

Corollary 9.8

If $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$ then the closure of some E_n contains an interval.