Nowhere Dense Sets

Note Title 8/20/2015

Definition (Also given in Problem 4.54)

A subset E of a metric space M is called <u>nowhere dense in M</u> if $\bar{E}$ contains no nonempty open sets.

Recalling that the interior of $\bar{E}$ is defined as

$$\operatorname{int}(\bar{E}) = \bigcup \{U : U \text{ is open, } U \subset \bar{E}\}$$

we see that E is nowhere dense iff $\operatorname{int}(\bar{E}) = \emptyset$.

(In fact, this is the definition Carothers gives in Problem 4.54.)

Since $\bar{E}$ does not contain any open sets, we must have the following: given any open set $U \subset M$ there exists $x \in U \cap \bar{E}^c$ (otherwise, we would have $U \subset \bar{E}$). This means $\bar{E}^c$ is dense in M. I.e., E is nowhere dense in M iff $\bar{E}^c$ is dense in M.

Examples

(a) $\overline{\mathbb{N}} = \mathbb{N}$, so $\overline{\mathbb{N}}^c = \mathbb{R} \setminus \mathbb{N}$, which is clearly dense in $\mathbb{R}$. We conclude that $\mathbb{N}$ is nowhere dense in $\mathbb{R}$. Likewise, we saw in Problem 2.22 that $\overline{\Delta}^c$ is dense in $[0,1]$, so Δ is nowhere dense in $\mathbb{R}$ (or $[0,1]$).

(b) Any finite union of nowhere dense sets will be nowhere dense, but a countable union of nowhere dense sets may not be nowhere dense. For example $\mathbb{Q}$ can be regarded as a countable union of single elements, and it is dense in $\mathbb{R}$. (So certainly not nowhere dense.)

(c) While $\mathbb{N}$ is nowhere dense in $\mathbb{R}$, it is of course not nowhere dense in $\mathbb{N}$. (Any set is dense in itself.)

(d) The designation "not nowhere dense" is <u>not</u> the same as dense. For example, $(0,1)$ is not nowhere dense in $\mathbb{R}$ (because $\mathrm{int}(\overline{(0,1)}) = (0,1) \neq \emptyset$), but it is certainly not dense in $\mathbb{R}$ either. As Carothers puts it, nowhere dense means "not even a little bit dense."