The Baire Category Theorem

Note Title 8/21/2015

Definitions

Let A be a subset of a metric space M.

(i) We say A is <u>of the first category in M</u> if it can be written as a countable union of sets, each of which is nowhere dense in M. Such sets are sometimes called "meager" or "sparse."

(ii) We say A is <u>of the second category in M</u> if it is not of the first category in M.
I.e., if A is of the second category in M then whenever we write $A = \bigcup_{n=1}^{\infty} E_n$, we must have that at least one of the E_n is <u>not</u> nowhere dense.

<u>Example</u>

We've seen in the previous lecture that $\mathbb{Q}$ is a first category set in $\mathbb{R}$. Corollary 9.8 tells us that $\mathbb{R} \setminus \mathbb{Q}$ is a second category set in $\mathbb{R}$, so we can view $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ as our model examples of first and second category sets.

Theorem 9.11 (The Baire Category Theorem)

For any complete metric space (M,d), if (G_n) is a sequence of dense open sets in M, then $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$; in fact $\bigcap_{n=1}^{\infty} G_n$ is dense in M.

It follows that (M,d) is of the second category in itself. That is, if we write $M = \bigcup_{n=1}^{\infty} E_n$, then the closure of some E_n contains an open ball (i.e., at least one of the E_n is <u>not</u>

nowhere dense in M).

Note on the proof

The first part of the statement is a straightforward generalization of the BCT on $\mathbb{R}$, and can be proved in the same way, with open balls used in place of open intervals. The details are left to Problem 9.26.

Let's check that the second part of our statement follows from the first.

Assume not; i.e., that the second part does not hold. This means we can write

$$M = \bigcup_{n=1}^{\infty} E_n$$

where for each $n \in \mathbb{N}$, E_n is nowhere dense in M. Notice that

$$M = \bigcup_{n=1}^{\infty} \overline{E_n},$$

and by De Morgan's Laws

$$\emptyset = M^c = \bigcap_{n=1}^{\infty} \overline{E}^c.$$

But we've seen that if E_n is nowhere dense in M, then $\overline{E}_n^c$ is dense in M, so by the first part of our theorem statement we have that $\bigcap_{n=1}^{\infty} \overline{E}_n^c$ is dense in M. We just said that this intersection should be the empty set, and this is our contradiction. "$\square$"

As with the BCT on $\mathbb{R}$, this asserts the existence of a non-empty set $\bigcap_{n=1}^{\infty} G_n$ (with certain properties), and in this way the BCT is often used in existence proofs.