Note Title 6/8/2015

The Nested Interval Theorem

<u>Theorem 1.5</u> (Nested Interval Theorem)

If (I_n) is a sequence of closed, bounded intervals in $\mathbb{R}$, with

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ ($\emptyset$ is the "empty set")

If, in addition, $\text{length}(I_n) \to 0$ then

$\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.

For example, we might have $I_n = [-\frac{1}{n}, \frac{1}{n}]$, and the theorem asserts $\bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}]$ contains one point. What would it be? 0.

<u>Proof</u>

Write $I_n = [a_n, b_n]$, and notice that $I_n \supset I_{n+1}$ means $a_n \le a_{n+1} \le b_{n+1} \le b_n \ \forall n$.

This means the sequence of left endpoints is monotone increasing, while the sequence of right endpoints is monotone decreasing (both bounded), so

$$a = \lim_{n\to\infty} a_n = \sup_n a_n$$

$$b = \lim_{n\to\infty} b_n = \inf_n b_n$$

both exist.

This means
$$\bigcap_{n=1}^{\infty} [a_n, b_n] = [a, b].$$

One easy way to see this is to observe that if $\underline{a_n \le x \le b_n \ \forall n}$ then $\underline{a \le x \le b}$, and conversely if $a \le x \le b$ then $a_n \le x \le b_n$ for all n. Finally, if $b_n - a_n = \text{length}(I_n) \to 0$ then $a = b$, and this is the only remaining point. $\square$