Note Title 6/8/2015

Geometric Series and Proposition 1.7

A geometric series has the form

$$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \cdots$$

Set $S_N = \sum_{n=1}^{N} r^n$ (the partial sum)

and notice that

$$S_N = r \sum_{n=1}^{N} r^{n-1} = r\left(S_N + 1 - r^N\right)$$

$$S_N(1-r) = -r^{N+1} + r$$

$$\Rightarrow S_N = \frac{-r^{N+1}+r}{1-r} = \frac{r-r^{N+1}}{1-r}$$

If $|r|<1$ then

$$\lim_{N\to\infty} S_N = \frac{r}{1-r}$$

That is: $$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

Proposition 1.7

Fix any integer $p \geq 2$, and let (a_n) be any sequence of integers satisfying $0 \leq a_n \leq p-1$ $\forall n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ converges to a number in $[0,1]$.

Proof

Since $a_n \geq 0$ the partial sums

$$S_N = \sum_{n=1}^{N} \frac{a_n}{p^n}$$

form a monotone increasing sequence. If we can show that that the sequence (S_N) is bounded by 1 we'll get the conclusion by Theorem 1.4.

$$\sum_{n=1}^{N} \frac{a_n}{p^n} \leq \sum_{n=1}^{N} \frac{p-1}{p^n} \leq (p-1) \sum_{n=1}^{\infty} \frac{1}{p^n}$$

$$= (p-1) \sum_{n=1}^{\infty} \left(\frac{1}{p}\right)^n = (p-1) \frac{1/p}{1 - 1/p} \frac{p}{p}$$

$$= \cancel{(p-1)} \frac{1}{\cancel{p-1}} = 1 \qquad \square$$