Some Fine Print about p-adic Expansions

For example, we can write

$$\frac{1}{2} = .5$$

or

$$\frac{1}{2} = .4\overline{9}$$

Our p-adic expansion is designed to get the second of these

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

This problem arises when

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n + 1}{p^n}$$

for n.

In this case (by finding a common denominator) we can write

$$x = \frac{I}{p^n}$$

for some integer I.

At the stage of choosing a_{n+1} we have

$$\frac{a_1}{p} + \frac{a_2}{p^2} + \dots + \frac{a_n}{p^n} + \frac{a_{n+1}}{p^{n+1}} < x$$

Now $a_{n+1} = p$ would give us equality, but this choice is not allowed, so we take $a_{n+1} = p-1$. This gives

$$\frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n} + \frac{p-1}{p^{n+1}} + \frac{a_{n+2}}{p^{n+2}} < x$$

In this case, again $a_{n+2} = p$ would give equality, but this is not allowed, so we take $a_{n+2} = p-1$. Proceeding in this way, we get the repeating decimal. I.e., we have two different representations:

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n + 1}{p^n}$$

and

$$x = \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n} + \sum_{k=n+1}^{\infty} \frac{p-1}{p^k}.$$

If $p = 10$, the sum is

$$\sum_{k=n+1}^{\infty} \frac{9}{10^k}$$

repeated decimal

which corresponds with repeating 9's.