Euler's Number

Note Title 6/9/2015

Proposition 1.10

(i) $\left(1+\frac{1}{n}\right)^n$ is strictly increasing

(ii) $\left(1+\frac{1}{n}\right)^{n+1}$ is strictly decreasing

(iii) $2 \leq \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n}\right)^{n+1} \leq 4$

(iv) Both sequences (from (i) and (ii)) converge to the same limit, which we designate by e and call Euler's number. Finally, $2 < e < 4$.

Proof

For (i) we need to show

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}$$

which is equivalent to

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} > 1.$$

We compute:

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \left(1+\frac{1}{n}\right)\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n(n+1)+n}{n(n+1)+n+1}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{(n+1)^2-1}{(n+1)^2}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1}$$

$$> \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right)$$

Bernoulli's inequality

$$= \frac{\cancel{n+1}}{\cancel{n}} \cdot \frac{\cancel{n}}{\cancel{n+1}} = 1$$

(ii) is almost identical to (i); see p. 9 in Carothers.

For (iii), we've seen that $\left(1+\frac{1}{n}\right)^n$ is increasing, and for $n=1$ we get 2. Likewise, $\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing, and for $n=1$ we get 4.

For (iv), the sequence $\left(1+\frac{1}{n}\right)^n$ is strictly increasing, and bounded above (by 4), so $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ converges. We call this limit e. Then

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1\cdot e = e.$$

Last, since $\left(1+\frac{1}{n}\right)^n$ is increasing, we can get a lower bound on e by evaluating at any n. E.g., for $n=2$

$$e > \left(1+\frac{1}{2}\right)^2 = \frac{9}{4} > 2.$$

Likewise, we can get an upper bound from $\left(1+\frac{1}{n}\right)^{n+1}$. $\square$

Euler's number is approximately

$$e = 2.71828....$$