Bolzano-Weierstrass Theorem

Note Title 6/10/2015

In order to work with liminf and limsup it's useful to have an ε characterization. To do this, recall that if (a_n) is bounded we have

$$M = \limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup\{a_k : k \geq n\}\right).$$

We can characterize M as follows:

M is the number so that given any $\varepsilon > 0$ there is an integer N so that

$$n \geq N \implies M - \varepsilon < \sup\{a_k : k \geq n\} < M + \varepsilon.$$

This means a_n cannot be greater that $M + \varepsilon$ for any $n \geq N$, and we can never run out of indices so that $a_n > M - \varepsilon$ (we must have an infinite number of such indices).

We use this as a useful characterization of limsup, which Carothers indicates with * on p. 12.

For every $\varepsilon > 0$ we have $a_n < M + \varepsilon$ * for all but finitely many n, and $a_n > M - \varepsilon$ for infinitely many n.

For a sequence (a_n) we have values

$$a_1, \; a_2, \; a_3, \; \ldots$$

and a subsequence would look something like

$$a_3, \; a_7, \; a_{11}, \; \ldots$$

We'll generally express these as

$$a_{n_1}, \; a_{n_2}, \; a_{n_3},$$

so for our example $n_1 = 3$, $n_2 = 7$, $n_3 = 11$ etc.

We write $(a_{n_k}) \subset (a_n)$ because of course the set of values in the subsequence is a subset of values from the original sequence.

In Problem 1.27 we'll use this to show that every sequence of real numbers (a_n) has a subsequence $(a_{n_k}) \subset (a_n)$ that converges to $\limsup_{n\to\infty} a_n$ (with $\pm$ allowed).

If we focus only on bounded sequences, this gives:

<u>Theorem 1.11</u> (Bolzano-Weierstrass)

Every bounded sequence of real numbers has a convergent subsequence.