Note Title 6/10/2015

Cauchy Sequences

Definition

We say (a_n) is Cauchy if given any $\varepsilon > 0$ there exists an integer N sufficiently large so that

$$n, m > N \Rightarrow |a_n - a_m| < \varepsilon.$$

Corollary 1.13

Every Cauchy sequence of real numbers converges.

Proof

We've seen in Problems 1.14 and 1.15 that Cauchy sequences are bounded, and we know that bounded sequences have convergent subsequences. Finally, from Problem 1.15 we know that a Cauchy sequence with a convergent subsequence converges. $\square$

Corollary 1.14

Every bounded sequence of real numbers has a Cauchy subsequence.

Proof

We know that every bounded subsequence has a convergent subsequence, and that every convergent sequence (subsequence here) is Cauchy. □

Proposition 1.15

If (a_n) is bounded, and if

$$\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n$$

then (a_n) converges and $\lim_{n \to \infty} a_n = \limsup_{n \to \infty} a_n$.

Proof

Let $a = \limsup_{n \to \infty} a_n$, and let $\varepsilon > 0$ be given.

From characterization $*$ there exists N_1 so that $a_n > a - \varepsilon$ $\forall$ $n \geq N_1$ (this is the liminf version of $*$), and there exists N_2 so that $a_n < a + \varepsilon$ $\forall$ $n \geq N_2$ (this is the limsup version of $*$). This implies for $n \geq \max(N_1, N_2)$ we have $|a - a_n| < \varepsilon$, and this means that (a_n) converges to a. $\square$

Theorem 1.16

A sequence of real numbers converges if and only if it is Cauchy.

This follows immediately from Problem 1.14 and Corollary 1.13.