# Cardinality of $P(\mathbb{N})$

Note Title 6/13/2015

Notice that if $A$ and $B$ are finite sets with $\text{card}(A) = m$ and $\text{card}(B) = n$ then

$$\text{card}(A \times B) = mn \quad (= \text{card}(A)\,\text{card}(B))$$

We extend this to the case of infinite sets by using it to define what we mean by cardinal multiplication. For example, the fact that $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$

leads us to define

$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

I.e.,

$$\operatorname{card}(\mathbb{N}) \cdot \operatorname{card}(\mathbb{N}) = \operatorname{card}(\mathbb{N} \times \mathbb{N})$$

Consider the collection of all sequences of 0's and 1's. First, let's check that this is uncountable. Recall that using binary numbers we can represent any number in terms of 0's and 1's.

For example,

$$7_{10} = 111_2 = 1\times 2^2 + 1\times 2^1 + 1\times 2^0$$

$$\frac{3}{4} = .11_2 = 1\times 2^{-1} + 1\times 2^{-2}$$

We know from Proposition 1.8 that we can write any $x \in [0,1]$ as

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$$

for a sequence $(a_n)$ of 0's and 1's. Since we can get all values in $[0,1]$ this way (i.e., we say the map from 0-1 sequences to $[0,1]$ is onto) the cardinality of the 0-1 sequences is at least as large as the cardinality of $[0,1]$. The map is not quite 1-1 because of repeating decimals, but we've seen there are only a countable number of those. If we take a countable number away from a set of size $c$, we still have a set of size $c$. (See Problem 2.17.) So the cardinality of the 0-1 sequences is $c$.

Next, we check that the set of 0-1 sequences is equivalent to $P(\mathbb{N})$. To see this, note that we can characterize any subset of $\mathbb{N}$ with 0's and 1's as follows: we put a 1 if the integer is included in the set, and a 0 if it is not. For example, suppose our set is $\{1, 2, 7\}$. We characterize this by

$$1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ \cdots$$

the rest are 0's.

We see then that $P(\mathbb{N})$ has the same cardinality of the 0-1 sequences which have the same cardinality as $[0,1]$. Recalling

$\mathrm{card}([0,1]) = c$, we have

$$\mathrm{card}(\mathcal{P}(\mathbb{N})) = c.$$

In analogy with our result that the cardinality of the power set of a set with n elements has $2^n$ elements, we write

$$c = 2^{\aleph_0}$$

Carothers mentions that for this reason the base 2 is sometimes used to designate power sets. That is, we write

$$\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$$

and more generally $\mathcal{P}(A) = 2^{A}$.