Countable Unions of Countable sets

Note Title 6/12/2015

Terminology

We'll say a set is countable if it's equivalent to $\mathbb{N}$ or some subset of $\mathbb{N}$.

Theorem 2.6

The countable union of countable sets is countable; that is, if A_i is countable for $i = 1, 2, 3, \ldots$, then $\bigcup_{i=1}^{\infty} A_i$ is countable.

Proof

Recall our proof that $\mathbb{N} \times \mathbb{N}$ is equivalent $\mathbb{N}$:

$$
\begin{matrix}
\overset{1}{(1,1)}, & \overset{3}{(1,2)}, & \overset{6}{(1,3)}, & \dots \\
\overset{2}{(2,1)}, & \overset{5}{(2,2)}, & (2,3), & \dots \\
\overset{4}{(3,1)}, & (3,2), & (3,3), & \dots \\
\vdots & & &
\end{matrix}
$$

Likewise, given any countable sets $A_1, A_2, \dots$ we can arrange the entries as follows:

$$\begin{array}{llllll}
A_1: & \underset{1}{a_{1,1}} & \underset{3}{a_{1,2}} & \underset{6}{a_{1,3}} & \dots \\
A_2: & \underset{2}{a_{2,1}} & \underset{5}{a_{2,2}} & a_{2,3} & \dots \\
A_3: & \underset{4}{a_{3,1}} & a_{3,2} & a_{3,3} & \dots
\end{array}$$

We can now count all entries in $\bigcup_{i=1}^{\infty} A_i$ by the same diagonalization process as before.

Corollary 2.7

The rational numbers are countable.

Proof

This follows because the rational numbers are a countable union of countable sets. Notice that we can arrange the positive rational numbers as:

$$\begin{array}{cccc}
\frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \dots \\
\frac{1}{2}, & \frac{2}{2}, & \frac{3}{2}, & \dots \\
\frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \dots
\end{array}$$

We have the same number of negative rational numbers as positive ones, and also 0. $\square$

Recall that we found earlier that for p-adic expansions $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$ we get two different decimal representations if and only if $x = \frac{q}{p^n}$ for some integer q, and some $n = 0, 1, 2, \ldots$.

Notice that for each integer q there are at most a countable number of possible n.

So there are precisely a countable union of countable sets total. Our theorem asserts, then, that the number of troublesome expansions is countable.