

Countable Unions of Countable sets

Note Title

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Terminology

We'll say a set is countable if it's equivalent to \mathbb{N} or some subset of \mathbb{N} .

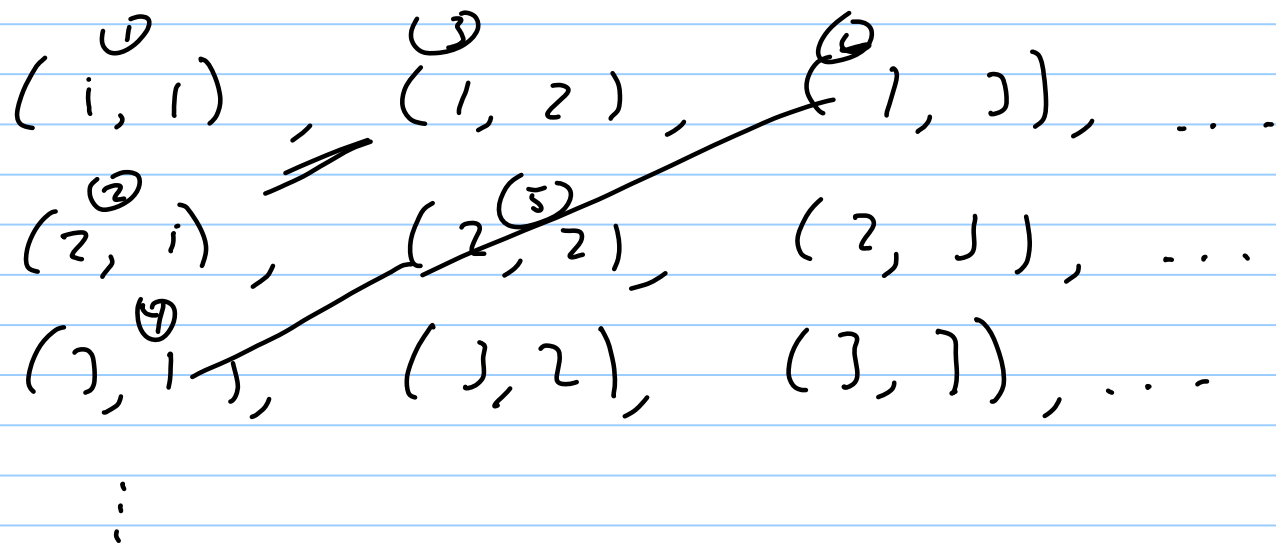
Theorem 2.6

The countable union of countable sets is countable; that is, if A_i is countable for $i = 1, 2, 3, \dots$, then $\bigcup_{i=1}^{\infty} A_i$ is countable.

Proof

Recall our proof that $\mathbb{N} \times \mathbb{N}$ is equivalent

\mathbb{N} :



Likewise, given any countable sets A_1, A_2, \dots

we can arrange the entries as follows:

$$\begin{array}{l} A_1: \quad a_{1,1} \quad a_{1,2} \quad a_{1,3} \quad \dots \\ A_2: \quad a_{2,1} \quad a_{2,2} \quad a_{2,3} \quad \dots \\ A_3: \quad a_{3,1} \quad a_{3,2} \quad a_{3,3} \quad \dots \end{array}$$

We can now count all entries in $\bigcup_{i=1}^{\infty} A_i$
by the same diagonalization process as before.

Corollary 2.7

The rational numbers are countable.

Proof

This follows because the rational numbers are a countable union of countable sets.

Notice that we can arrange the positive rational numbers as:

$$\begin{array}{cccc} \frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \dots \\ \frac{1}{2}, & \frac{2}{2}, & \frac{3}{2}, & \dots \\ \frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \dots \end{array}$$

We have the same number of negative rational numbers as positive ones, and also 0. \square

Recall that we found earlier that for p -adic expansions $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$ we get two different decimal representations if and only if $x = \frac{q}{p^n}$ for some integer q , and some $n = 0, 1, 2, \dots$.

Notice that for each integer q there are at most a countable number of possible n .

So there are precisely a countable union of countable sets total. Our theorem asserts, then, that the number of troublesome expansions is countable.