Note Title 6/12/2015

Uncountable Sets

Definition

Any infinite set that is not countable is called uncountable.

Theorem 2.9

$\mathbb{R}$ is uncountable

Proof

In fact, we'll show that $(0,1)$ is uncountable, and the same would be true for any interval (a,b).

We argue by contradiction, assuming we can list all elements of $(0,1)$ as a sequence (a_n).

E.g.:

$a_1 = .\,3\;1\;5\;7\;2\ldots$

$a_2 = .\,0\;4\;2\;6\;8\ldots$

$a_3 = .\,9\;1\;5\;3\;6\ldots$

$a_4 = .\,7\;5\;9\;9\;7\ldots$

$\vdots$

Carothers suggests using 3 if the entry is not 3, and 5 if it is.
For our example, we would get —

$$x = .5333\ldots$$

By construction x <u>cannot</u> be in the list, and this is a contradiction to the assumption that we could enumerate the elements of $(0,1)$. $\square$

This is called Cantor's diagonalization argument. Notice that the following corollary is an immediate consequence:

<u>Corollary 2.10</u>

$\mathbb{R} \setminus \mathbb{Q}$ (i.e., the irrational numbers) is uncountable.