Note Title 6/12/2015

Cantor's Theorem

Theorem 2.12 (Cantor's Theorem)

Let A be any set. No map $F: A \to P(A)$ can be onto. I.e., it's impossible to set up a 1-1 correspondence between A and $P(A)$. This means A and $P(A)$ cannot be equivalent.

Proof

Given any map $F : A \to P(A)$, define

$$B := \{x \in A : x \notin F(x)\}.$$

Since B is a subset of A, $B \in P(A)$.

We claim that $B \neq F(y)$ for any $y \in A$. This will complete the proof because it means B cannot be obtained as a map from A.

Suppose $B = F(y)$

and notice that either:

$$y \in F(y) = B \implies y \notin F(y) \rightarrow\leftarrow$$

or

$$y \notin F(y) = B \implies y \in F(y) \rightarrow\leftarrow \quad \square$$

Cardinality will refer to our measure of set size.

We'll denote the cardinality of a set A card(A). For a finite set of n elements

$$\operatorname{card}(A) = n.$$

For a countably infinite set, we'll write

$$\operatorname{card}(A) = \underline{\underline{\aleph_0}}$$

This letter $\aleph$ is the Hebrew letter aleph, so this is read "aleph naught," "aleph null," or "aleph zero."

For an uncountable set the size of $\mathbb{R}$ we write

$$\mathrm{card}(\mathbb{R}) = c$$

(c is for continuum).

We would like cardinality to give a reasonable ordering of sets in at least the following way:

if $\mathrm{card}(A) \leq \mathrm{card}(D)$ and $\mathrm{card}(D) \leq \mathrm{card}(A)$, then we should have $\mathrm{card}(A) = \mathrm{card}(D)$. We'll prove that this is true in the next lecture.