P-adic Characterization of the Cantor Set

Theorem 2.15

$x \in \Delta$ if and only if x can be expressed as

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

where each a_n is either 0 or 2.

Proof

We know from Proposition 1.8 that each $x \in [0,1]$

can be expressed as

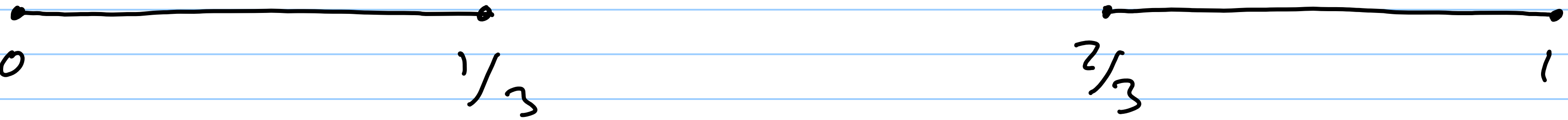
$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

for a sequence of integers (a_n) taking values $0, 1, 2$. I.e., these are the base 3 decimals

$$x = .a_1 a_2 a_3 \ldots \quad (\text{base } 3)$$

In order to connect this with the Cantor set, let's think about I_1:

I_1:



If $x \in [0, 1/3]$ then $a_1 = 0$, with the one exception that $x = 1/3$ can be expressed either as $x = .1$ or $x = .0\overline{2}$. This means that every $x \in \Delta \cap [0, 1/3)$ can be expressed with $a_1 = 0$. On the other hand, if $x \in [2/3, 1)$ then $a_1 = 2$ with the one exception that $x = 2/3$ can be expressed as $x = .2$ or $x = .1\overline{2}$. This means that every $x \in \Delta \cap [2/3, 1]$ can be expressed with $a_1 = 2$. We have:

If $x \in [0, 1/9]$ then $a_2 = 0$ with the one exception that $x = 1/9$ can be expressed either as $x = .01$ or $x = .00\overline{2}$. This means that every $x \in \Delta \cap [0, 1/9]$ can be expressed with $a_1 = 0$, $a_2 = 0$.

Continuing this way, we see that every element in Δ can be expressed with a_n

taking values only 0 and 2. $\square$

For example, this theorem tells us that $.020\overline{02}_3$ must be in Δ. But this is

$$\sum_{n=1}^{\infty} \frac{2}{9^n} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n$$

$$= 2\,\frac{1/9}{1 - 1/9} = \frac{1}{4}.$$

So $1/4 \in \Delta$.