A Second Proof that card(Δ) = c

Note Title 6/14/2015

Corollary 2.16

The Cantor set is equivalent to $[0,1]$.

Proof

We've seen that the Cantor function $f: \Delta \to [0,1]$ is onto, and also that it's non-decreasing:

$$x < y \implies f(x) \leq f(y).$$

The one slight problem we have with the Cantor function is that it's not 1-1. However, we'll check in Problem 2.2C that if $x < y$ then $f(x) = f(y)$ if and only if x and y are endpoints of a discarded middle third. Recall that these endpoints are easy to enumerate:

$$0,\ 1,\ \underbrace{\frac{1}{3},\ \frac{2}{3}}_{\text{step 1}},\ \underbrace{\frac{1}{9},\ \frac{2}{9},\ \frac{7}{9},\ \frac{8}{9}}_{\text{step 2}},\ \ldots$$

So this set of values is countable.

Define a new function g that is the same as the Cantor function, except that we define it on the domain obtained by removing from Δ the right endpoints $2/3$, $2/9$, $8/9$, etc. (not including 1, which was not obtained by deleting a middle third). Let's denote this new domain $\tilde{\Delta}$.

We see that $g : \tilde{\Delta} \to [0,1]$ is 1-1 and onto, so $\tilde{\Delta}$ and $[0,1]$ are equivalent.

Finally, since we've only taken a countable set away from Δ to get $\tilde{\Delta}$ we know from Problem 2.17 that Δ and $\tilde{\Delta}$ are equivalent. We conclude then that Δ must be equivalent to $[0,1]$, because $\tilde{\Delta}$ is.