In the previous lecture we defined

$$\Delta - \Delta := \{ y - x : x, y \in \Delta \}.$$

Theorem

$$\Delta - \Delta = [-1, 1].$$

Proof

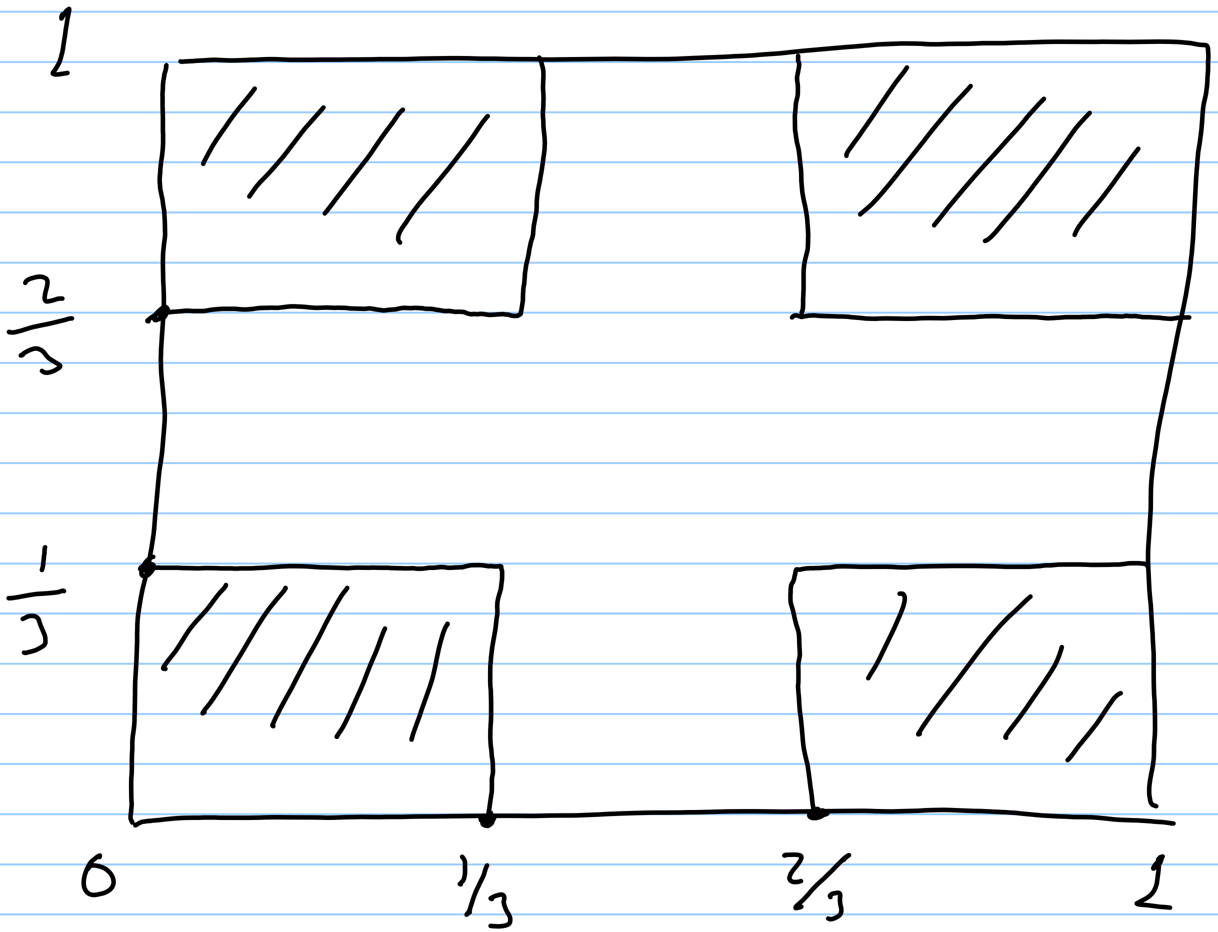
The statement is equivalent to the following: given any $b \in [-1, 1]$ there exist $x, y \in \Delta$ so that

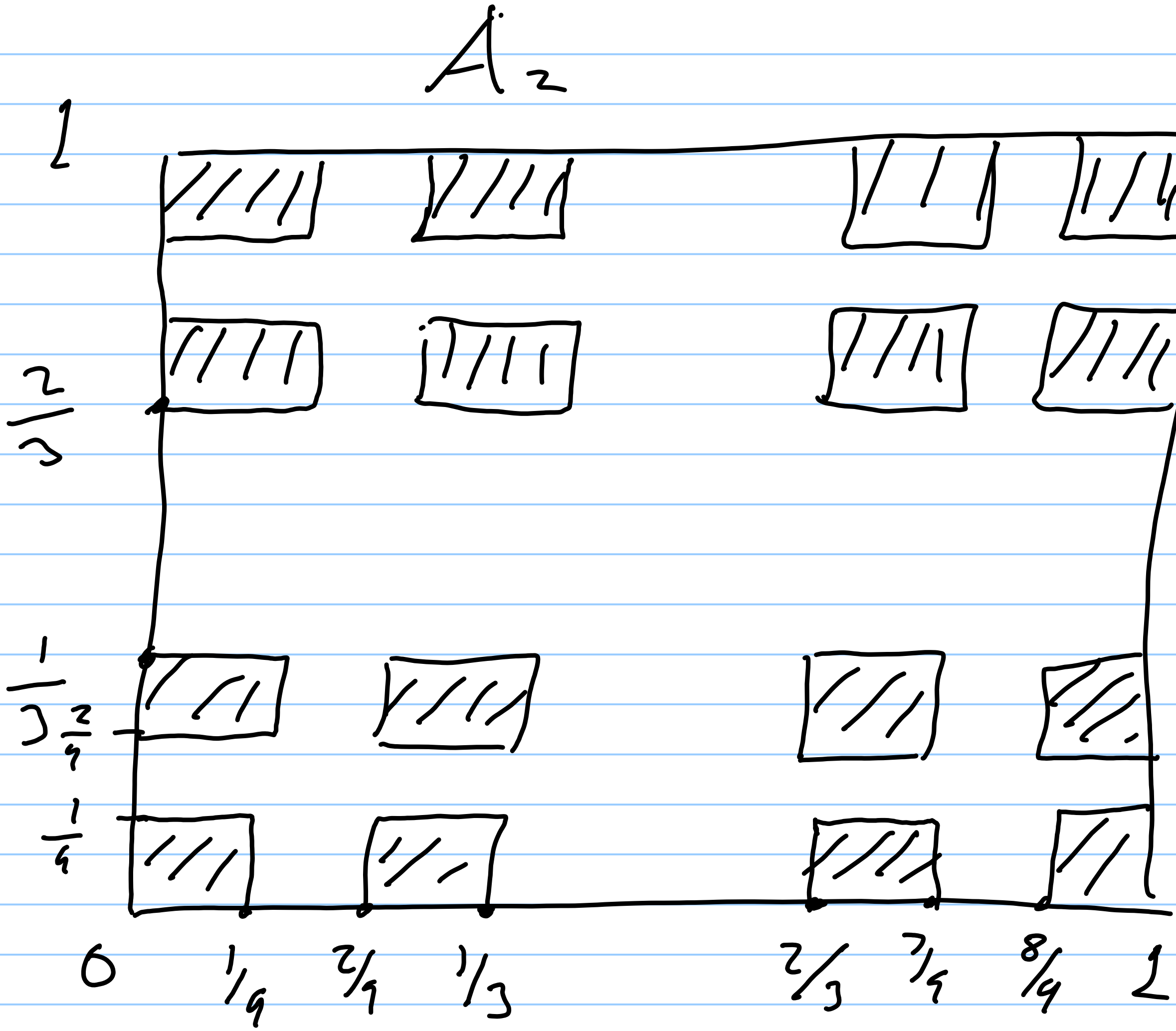
$$y - x = b.$$

Another way to say this is: given any $b \in [-1, 1]$ the line $y = x + b$ must pass through the set $\Delta \times \Delta$.

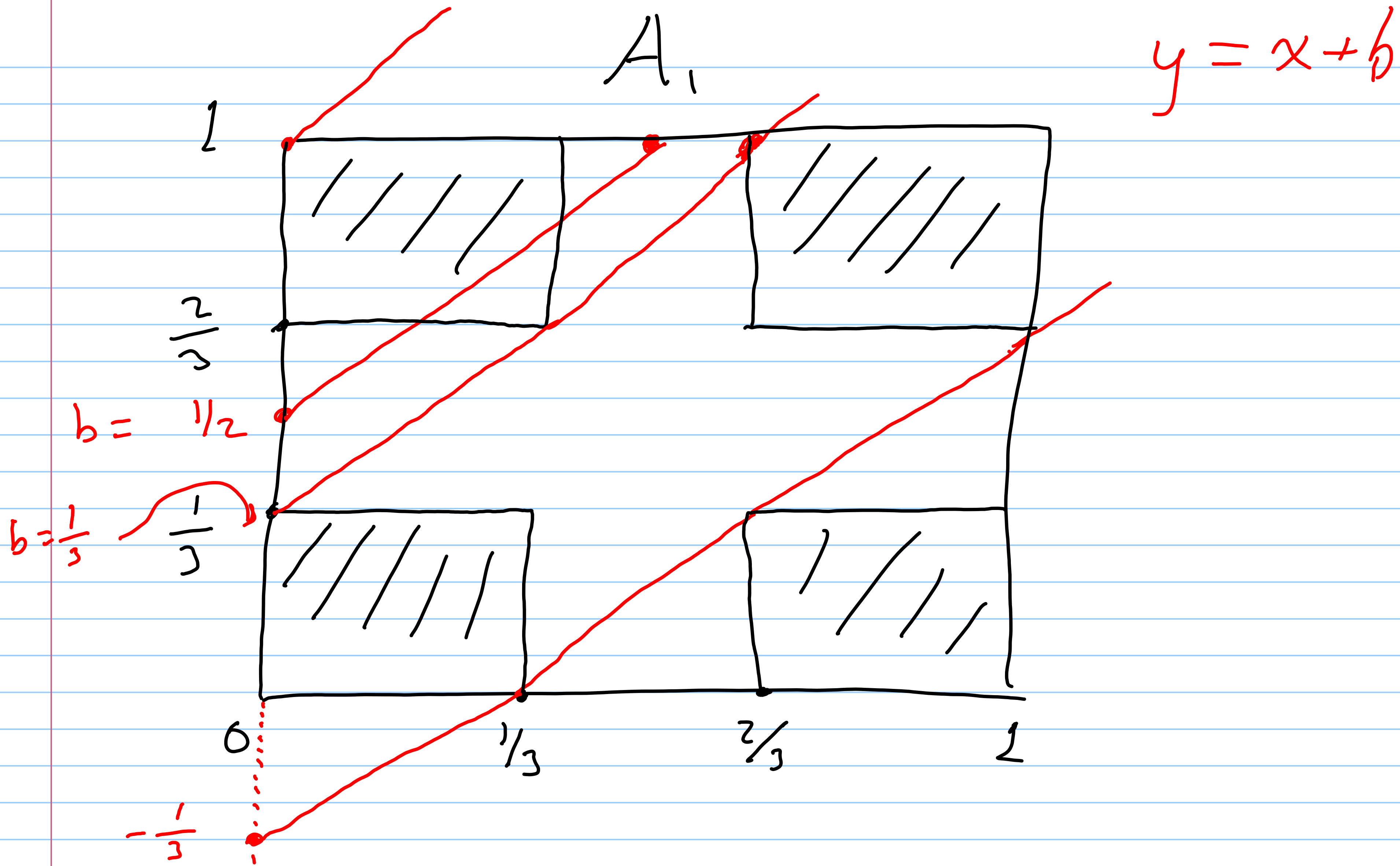
We can see this geometrically by constructing $\Delta \times \Delta$, which is interesting in its own right. We begin with a full square and remove the rectangles $(\frac{1}{3}, \frac{2}{3}) \times [0,1]$ and $[0,1] \times (\frac{1}{3}, \frac{2}{3})$:

A_1

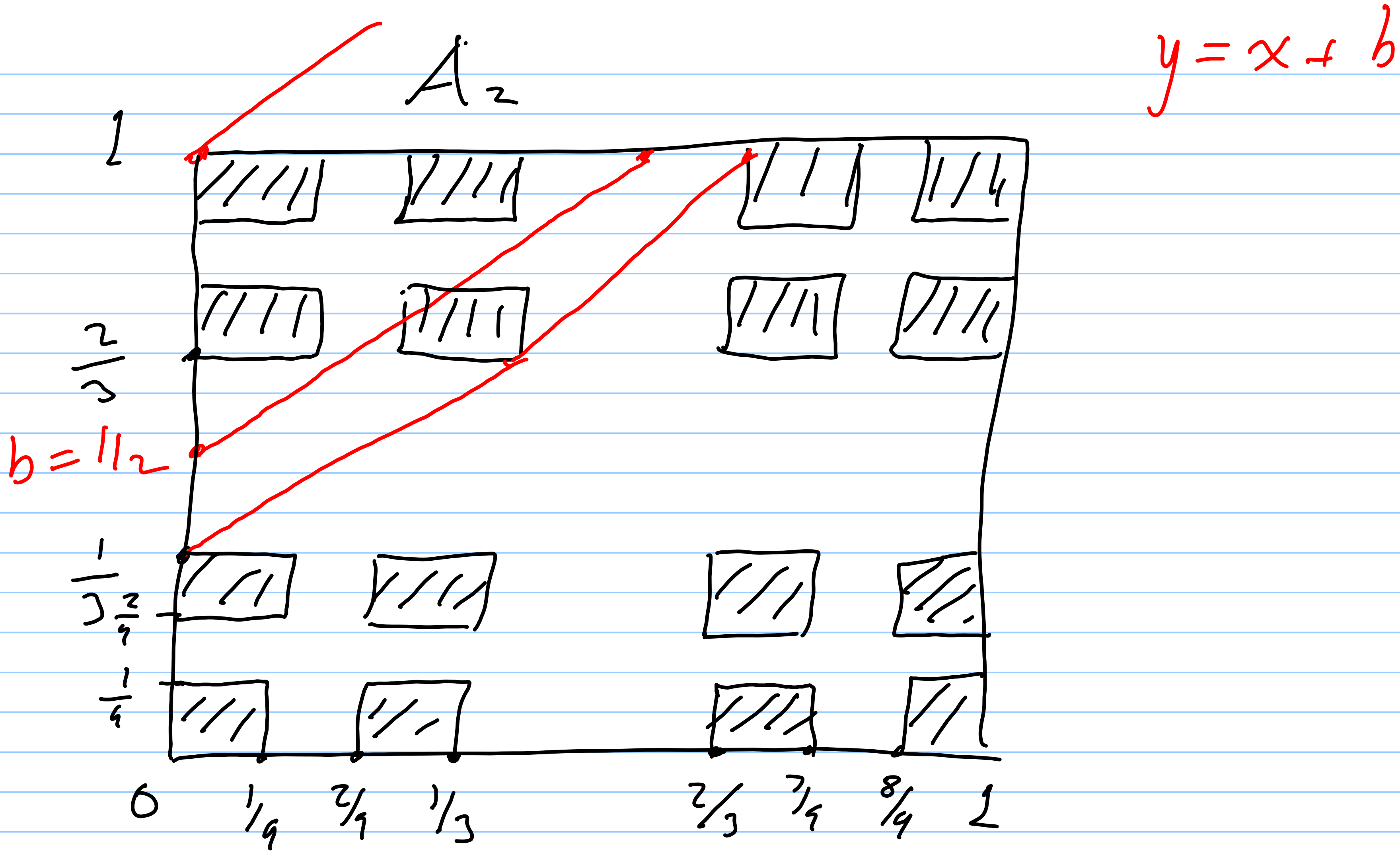



A_2
1
2/3
1/3
2/9
1/9
0
1/9
2/9
1/3
2/3
7/9
8/9
1

Starting with A_1, notice that if we draw the line $y = x + b$ through the square with any intercept $b \in [-1, 1]$ the line will intersect A_1 (i.e., will intersect at least one of the shaded boxes).



A_1
$y = x + b$
1
2/3
b = 1/2
b = 1/3
1/3
0
-1/3
1/3
2/3
1



A2
y = x + b
1
2/3
b = 1/2
1/3
2/9
1/9
0
1/9
2/9
1/3
2/3
7/9
8/9
1

Continuing in this way, we can see that the lines $y = x + b$ for any $b \in [-1, 1]$ intersect at least one shaded square in A_n $\forall\, n = 1, 2, \ldots$ We can conclude that these lines intersect

$$\Delta \times \Delta = \bigcap_{n=1}^{\infty} A_n .$$

This uses Theorem 7.11 in Chapter 7.