Familiar Metrics and Norms

Note Title 6/16/2015

As a start, recall that we compute the distance between $x, y \in \mathbb{R}$ as

$$d(x,y) = |x - y|$$

Here, d is an example of a <u>metric</u>.

Likewise, it's easy to measure the size of a point in $\mathbb{R}$: $\|x\| = |x|$. Here, $\|\cdot\|$ is an example of a norm.

As another familiar example, consider Euclidean space $\mathbb{R}^n$. The most common norm for $x \in \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

One possible metric for $\mathbb{R}^n$ is

$$d(x,y) = \|x - y\|.$$

As another collection of objects, consider the continuous functions on $[0,1]$, which we'll denote $C([0,1])$. In this case, several norms are common:

$$\|f\|_C = \max_{x\in[0,1]} |f(x)|$$

$$\|f\|_1 = \int_0^1 |f(x)|\,dx$$

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}$$

As above, each of these provides a metric, with $d(x,y) = \|x-y\|$.

Finally, we should note that not every metric arises this way. For example, we'll show in Problem 3.1 that

$$d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|$$

is a metric on $(0,\infty)$.