Metric Spaces

Note Title 6/17/2015

Definition

Given any set M, we say that a function d defined on $M \times M$ is a metric provided:

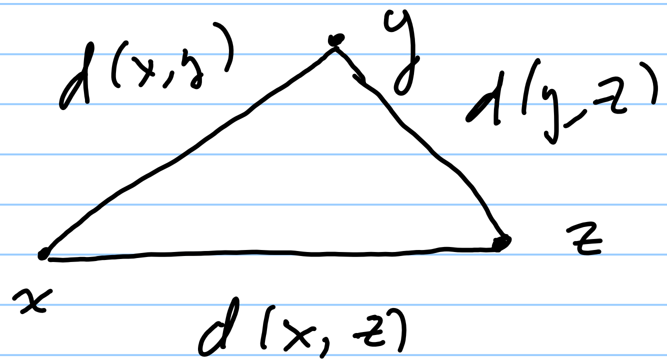
(i) $0 \leq d(x,y) < \infty \quad \forall\, x,y \in M$

(ii) $d(x,y) = 0$ if and only if $x = y$

(iii) $d(x,y) = d(y,x) \quad \forall\, x,y \in M$

(iv) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\, x,y,z \in M.$

Property (iv) is typically referred to as the triangle inequality.



We refer to a pair (M, d) of a space with a metric as a <u>metric space</u>.

Each of our examples from the previous lecture is a metric space.

Another important example is the discrete metric. The discrete metric can be defined on <u>any</u> space M and is:

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Let's check that this is a metric, by checking Properties (i) - (iv).
For (i), we have $0 \leq d(x,y) \leq 1$, so certainly $0 \leq d(x,y) < \infty$. We see that (ii) holds by definition, and likewise (iii) is immediate.

For (iv), if $x = y$ we have $d(x,y) = 0$, so the inequality must hold, and if $x \neq y$

then for any $z \in M$ we either have $z \neq x$ or $z \neq y$ (or both). This means that the right-hand side is at least one, and since the left-hand side must be less than or equal to 1, this gives the inequality.