Note Title 6/17/2015

Normed Vector Spaces

Definition

We say V is a vector space over $\mathbb{R}$ if $\forall\, x, y, z \in V$ and $\alpha, \beta \in \mathbb{R}$ we have:

1. $x + y \in V$
2. $\alpha x \in V$
3. $x + (y + z) = (x + y) + z$
4. $x + y = y + x$

5. $\exists\, 0 \in V$ so that $0 + x = x$

6. $\exists\, -x \in V$ so that $-x + x = 0$

7. $\alpha(\beta x) = (\alpha\beta)x$

8. $1x = x$

9. $\alpha(x+y) = \alpha x + \alpha y$

10. $(\alpha+\beta)x = \alpha x + \beta y$.

Definition

We say $\|\cdot\|$ is a norm on a vector space V provided:

(i) $0 \le \|x\| < \infty \quad \forall x \in V$

(ii) $\|x\| = 0$ if and only if $x = 0$

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$ for any $\alpha \in \mathbb{R}$, $x \in V$

(iv) $\|x + y\| \le \|x\| + \|y\| \quad \forall x, y \in V$.

A function satisfying all properties except (iii) is called a pseudonorm. As with metric spaces, we refer to pairings $(V, \|\cdot\|)$ as normed vector spaces. It's easy to check that any norm provides a metric:

$$d(x,y) = \|x-y\|.$$

One more useful example of a normed vector space is the extension of $\mathbb{R}^n$ (or $\mathbb{C}^n$) to infinite dimensions. We denote by ℓ_p (read: "little L - P") the space of infinite sequences (x_n) so that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

We also define l_∞ as the space of infinite sequences (x_n) so that

$$\sup_n |x_n| < \infty.$$

We'll show that

$p \geq 1$

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

and

$$\|x\|_\infty = \sup_n |x_n|$$

are norms on those important spaces.