Note Title 7/2/2015

Norms on ℓ_p Spaces, I

Lemma 3.5

Let $1 \le p < \infty$ and let $a, b \ge 0$. Then

$$(a+b)^p \le 2^p (a^p + b^p).$$

Consequently, $x + y \in \ell_p$ whenever $x, y \in \ell_p$.

Proof

It's clear that either $a+b \le 2a$ or $a+b \le 2b$, So $(a+b)^p \le (2a)^p + (2b)^p$

$$\Rightarrow \quad (a+b)^p \le 2^p(a^p + b^p).$$

For the second part, we have:

$$\sum_{n=1}^{\infty} |x_n + y_n|^p \le \sum_{n=1}^{\infty} 2^p\left(|x_n|^p + |y_n|^p\right)$$

$$= 2^p \sum_{n=1}^{\infty} |x_n|^p + 2^p \sum_{n=1}^{\infty} |y_n|^p < \infty. \qquad \square$$

Lemma 3.6 (Young's Inequality)

Let $1 < p < \infty$ and let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. (We say p and q are Hölder conjugates.) Then, for any $a, b \geq 0$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with equality occurring iff $a^{p-1} = b$.

Proof

If either $a = 0$ or $b = 0$ the inequality is obvious, so suppose $a, b \neq 0$. Notice that $\frac{1}{q} = 1 - \frac{1}{p} \implies q = \frac{1}{1 - 1/p} \frac{p}{p} = \frac{p}{p-1} \in (1, \infty)$.

We see that:

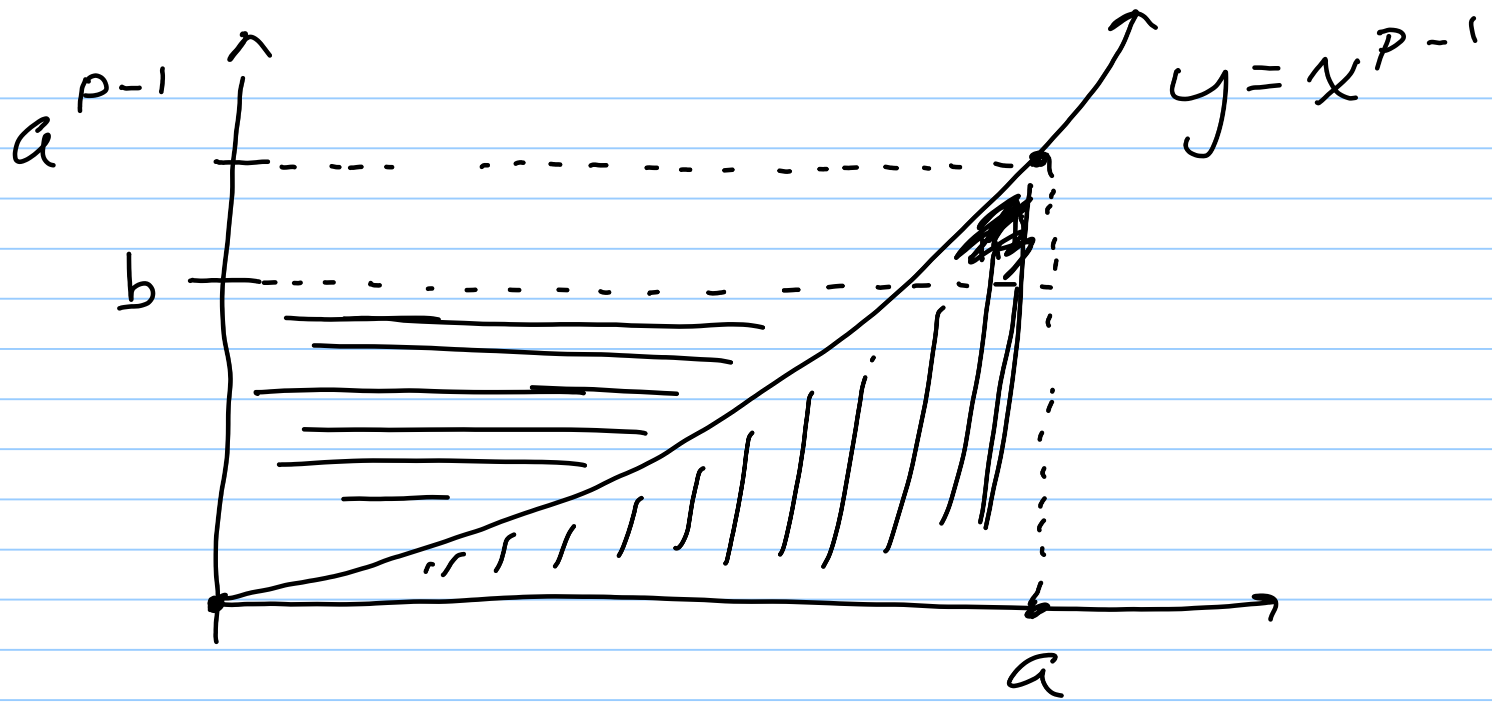
$$p - 1 = \frac{p}{q} = \frac{1}{q-1}$$

Let's set $f(x) = x^{p-1}$, and let's compute the inverse of $f(x)$.

$$y = x^{p-1} \implies x = y^{\frac{1}{p-1}} = y^{q-1}$$

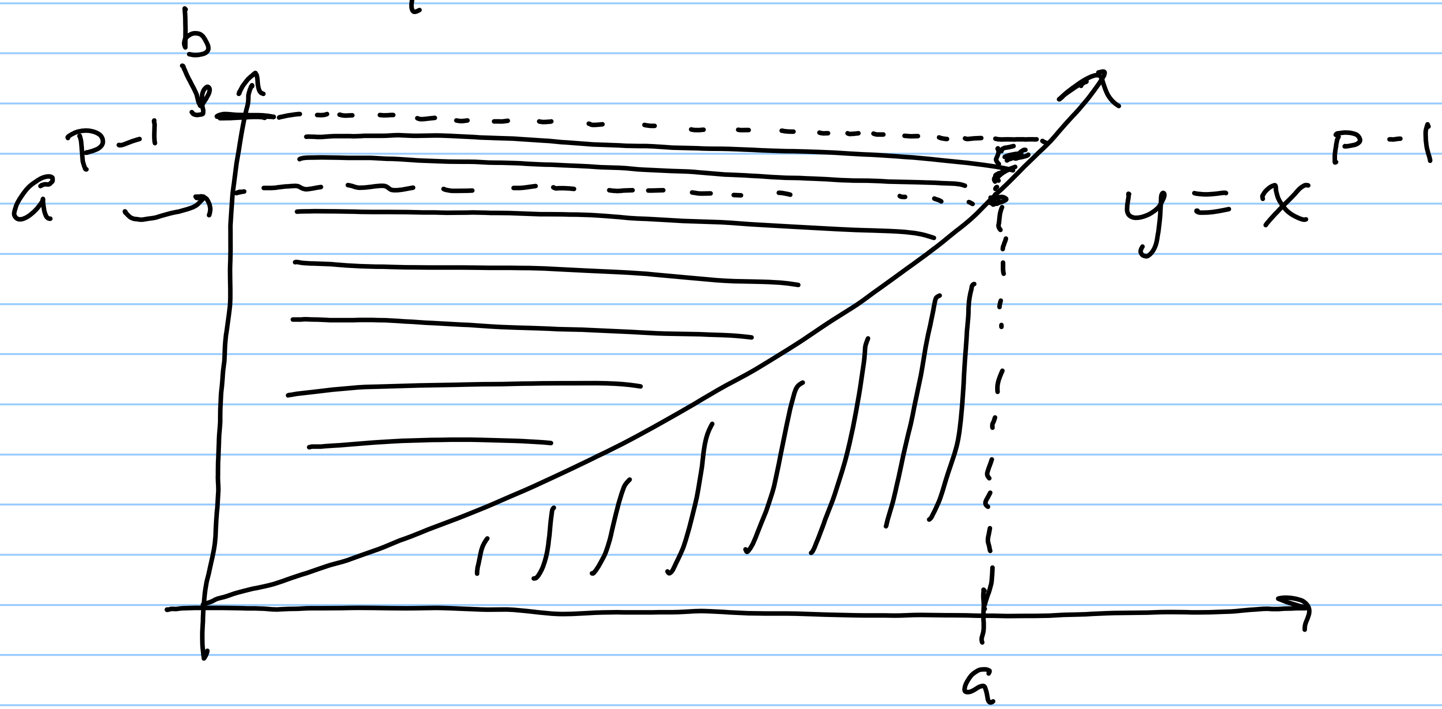
The inverse of f is $g(y) = y^{q-1}$.

Keeping in mind that $p > 1$, let's plot $y = x^{p-1}$ for $x \in [0, a]$:



The area indicated with vertical lines is $\int_0^a x^{p-1}\,dx$, and the area indicated with horizontal lines is $\int_0^b x^{q-1}\,dx$.

But we could also have $b > a^{p-1}$, in which case the picture would be:



In either case, the area obtained by integration is large than the area of the rectangle with sidelengths a and b. That is:

$$ab \le \int_0^a x^{p-1}\,dx + \int_0^b x^{q-1}\,dx$$

$$= \frac{x^p}{p}\Big|_0^a + \frac{x^q}{q}\Big|_0^b$$

$$= \frac{a^p}{p} + \frac{b^q}{q}.$$

Finally, we get equality precisely when those areas coincide, and this can only happen if $b = a^{p-1}$. $\square$