Open and Closed Sets in the Relative Metric

Note Title 7/16/2015

Proposition 4.13

Let (A, d) be a subset of a general metric space (M, d).

(i) A set $G \subset A$ is open in (A, d) iff $G = A \cap U$ for some open set U in (M, d).

(ii) A set $F \subset A$ is closed in (A,d) iff $F = A \cap C$, where C is some closed set in (M,d).

(iii) $cl_A(E) = A \cap cl_M(E)$ for any subset E of A.

closure in A closure in M

<u>Proof</u>

We'll prove (i), and leave the proofs of (ii) and (iii) to Problem 4.61.

For (i), we start with ($\Leftarrow$) and assume $G = A \cap \mathcal{U}$ for some open set $\mathcal{U}$ in (M,d). If $x \in G$, then $x \in \mathcal{U}$, so $B_\varepsilon^M(x) \subset \mathcal{U}$ for some $\varepsilon > 0$ ($\because$ $\mathcal{U}$ is open). But then

$$A \cap B_\varepsilon^M(x) \subset A \cap \mathcal{U} = G,$$

which is precisely what we need to conclude that G is open in (A,d).

For ($\Rightarrow$) we suppose $G \subset A$ is open in (A,d). Then for each $x \in G$ there is some ε_x so that

$$A \cap B^{M}_{\varepsilon_x}(x) \subset G.$$

Set $$U = \bigcup \{ B^{M}_{\varepsilon_x}(x) : x \in G \}.$$

This is an open set in (M,d), and notice that $A \cap U \subset G$ and also $G \subset A \cap U$ ($\because G \subset A$ and $G \subset U$), so $G = A \cap U$. $\square$