Open Sets

Note Title 7/12/2015

Definition (Open Set)

A set U in a metric space (M, d) is called an open set if U contains a neighborhood of each of its points; equivalently, if given any $x \in U$ there is an $\varepsilon > 0$ small enough so that $B_\varepsilon(x) \subset U$.

Some common examples include intervals $(a,b) \subset \mathbb{R}$, and balls $B_r(x) \subset \mathbb{R}^2$. The whole space M is always open, because M itself can be taken as a neighborhood, and $\emptyset$ is considered open because there are no points to check. In a discrete space, $B_1(x) = \{x\}$ can be taken as a neighborhood of x (or we can use any positive radius less than 1). So every subset of a discrete space is open.

Proposition 4.2

Let (M,d) be any metric space. For any $x \in M$ and $\varepsilon > 0$ the ball $B_\varepsilon(x)$ is an open set.

Proof

Take $y \in B_\varepsilon(x)$ so that $d(x,y) < \varepsilon$, and set

$$\delta := \varepsilon - d(x,y) > 0.$$

We claim $B_\delta(y) \subset B_\varepsilon(x)$. (This establishes the proposition.)

To see this, notice that if $d(y,z) < \delta$ (i.e., $z \in B_\delta(y)$) then

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \delta$$
$$= d(x,y) + \varepsilon - d(x,y) = \varepsilon \Rightarrow z \in B_\varepsilon(x). \quad \square$$