Closed Sets

Note Title 7/14/2015

Definition

A set F in a metric space (M,d) is said to be closed if its complement $F^c = M \setminus F$ is open.

Examples

The most common example of a closed set is an interval with its endpoints $[a,b]$. The complement of this set in $\mathbb{R}$ is

$$(-\infty, a) \cup (b, +\infty),$$

and it's easy to see that this set is open.

Any finite set of points is closed. To see this, notice that the complement of a single point $a \in \mathbb{R}$ is $(-\infty, a) \cup (a, +\infty)$, and this is an open set. Likewise, the complement of two points $a, b \in \mathbb{R}$, $a < b$, is

$$(-\infty, a) \cup (a, b) \cup (b, \infty),$$

and this is an open set. Similarly for more points.

Since the complement of $\emptyset$ is M and the complement of M is $\emptyset$, and since M and $\emptyset$ are both open, we see that M and $\emptyset$ are both closed. This means M and $\emptyset$ are both open and closed. Such sets are referred to as clopen. We noted earlier that every subset of a discrete space is open, and this means that every subset of a discrete space is closed. So every subset of a discrete space is clopen.

It follows from De Morgan's Laws that since an arbitrary union of open sets is open, an arbitrary intersection of closed sets is closed.

Likewise a <u>finite</u> union of closed sets is closed.

To see that an infinite union of closed sets is not necessarily closed, consider the sets

$$\left[-1+\frac{1}{n},\ 1-\frac{1}{n}\right]$$

and notice that

$$\bigcup_{n=1}^{\infty}\left[-1+\frac{1}{n},\ 1-\frac{1}{n}\right] = (-1,+1).$$

Finally, notice that many sets such as $(0,1]$ are neither open nor closed.