Note Title 7/15/2015

Sequential Characterization of Closure

Proposition 4.10

For a set A, $x \in \bar{A}$ iff $B_\varepsilon(x) \cap A \neq \emptyset$ $\forall\, \varepsilon > 0$.

Proof

For $(\Leftarrow)$ we note that if $B_\varepsilon(x) \cap A \neq \emptyset$ then certainly $B_\varepsilon(x) \cap \bar{A} \neq \emptyset$, and so by by Theorem 4.9 $x \in \bar{A}$.

For ($\Rightarrow$) suppose $x \in \bar{A}$ and take $\varepsilon > 0$. If $B_\varepsilon(x) \cap A = \emptyset$, then $A \subset B_\varepsilon(x)^c$, which is closed since $B_\varepsilon(x)$ is open. Thus $\bar{A} \subset B_\varepsilon(x)^c$ ($\because$ $\bar{A}$ is the smallest closed set containing A, and $B_\varepsilon(x)^c$ is a closed set containing A). But this containment (i.e., $\bar{A} \subset B_\varepsilon(x)^c$) is a contradiction $\because$ $x \in \bar{A}$ but of course $x \notin B_\varepsilon(x)^c$. $\square$

Corollary 4.11

For a set A, $x \in \bar{A}$ iff there is a sequence $(x_n) \subset A$ with $x_n \to x$.

Proof

This is immediate from Proposition 4.10, and the equivalence of (ii) and (iii) in Theorem 4.9. $\square$

Let's revisit $\overline{\mathbb{Q}}$. Notice that we can get every real number as the limit of a sequence of rational numbers, and this means (by Corollary 4.11) that $\overline{\mathbb{Q}} = \mathbb{R}$.