Continuity in Metric Spaces

Note Title 7/16/2015

Definition (Continuity at a point)

Let (M,d) and (N,ρ) be metric spaces. We say $f: M \to N$ is continuous at $x \in M$ if

$$\lim_{y \to x} f(y) = f(x).$$

If f is continuous at every point of M, we say f is continuous on M.

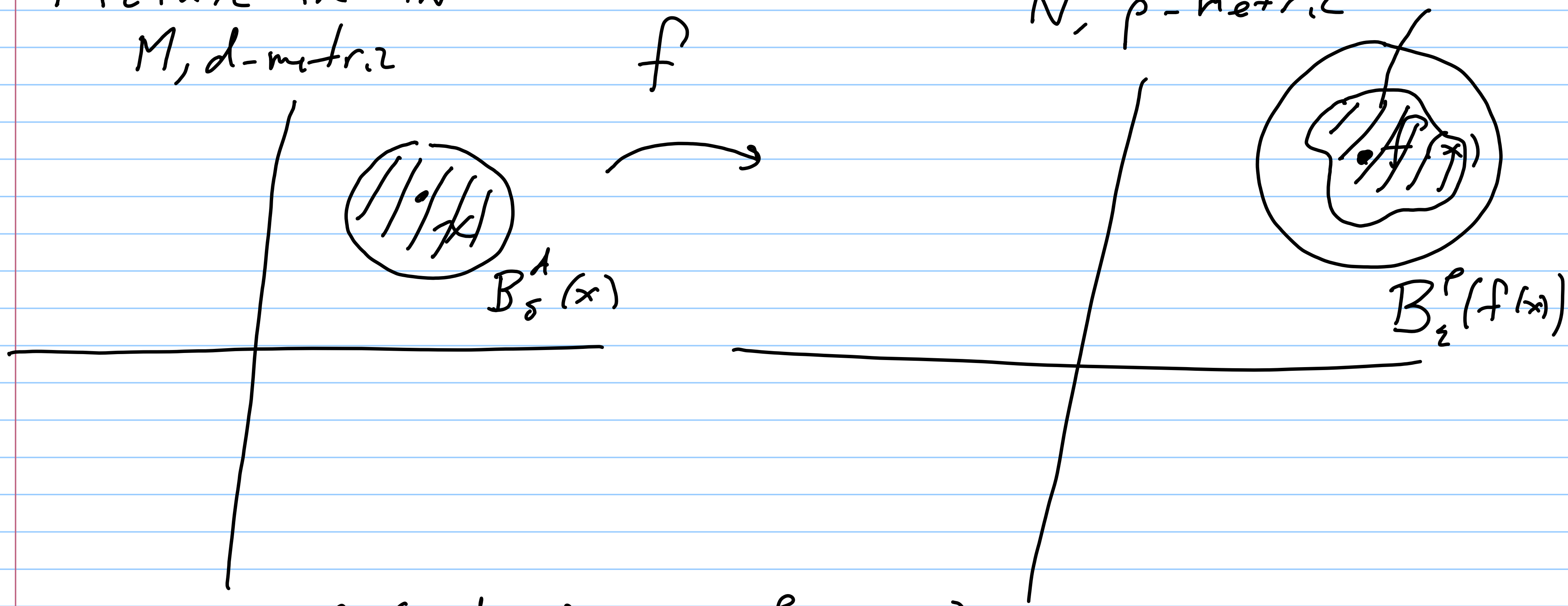
We can express continuity at a point x in the usual $\varepsilon - \delta$ sense as follows: given any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$d(x,y) < \delta \implies \rho(f(x), f(y)) < \varepsilon.$$

This says that f maps balls $B_\delta^d(x)$ (i.e., $d(x,y) < \delta$) into balls $B_\varepsilon^\rho(f(x))$ (i.e., $\rho(f(x), f(y)) < \varepsilon$):

$$f(B_\delta^d(x)) \subset B_\varepsilon^\rho(f(x)).$$

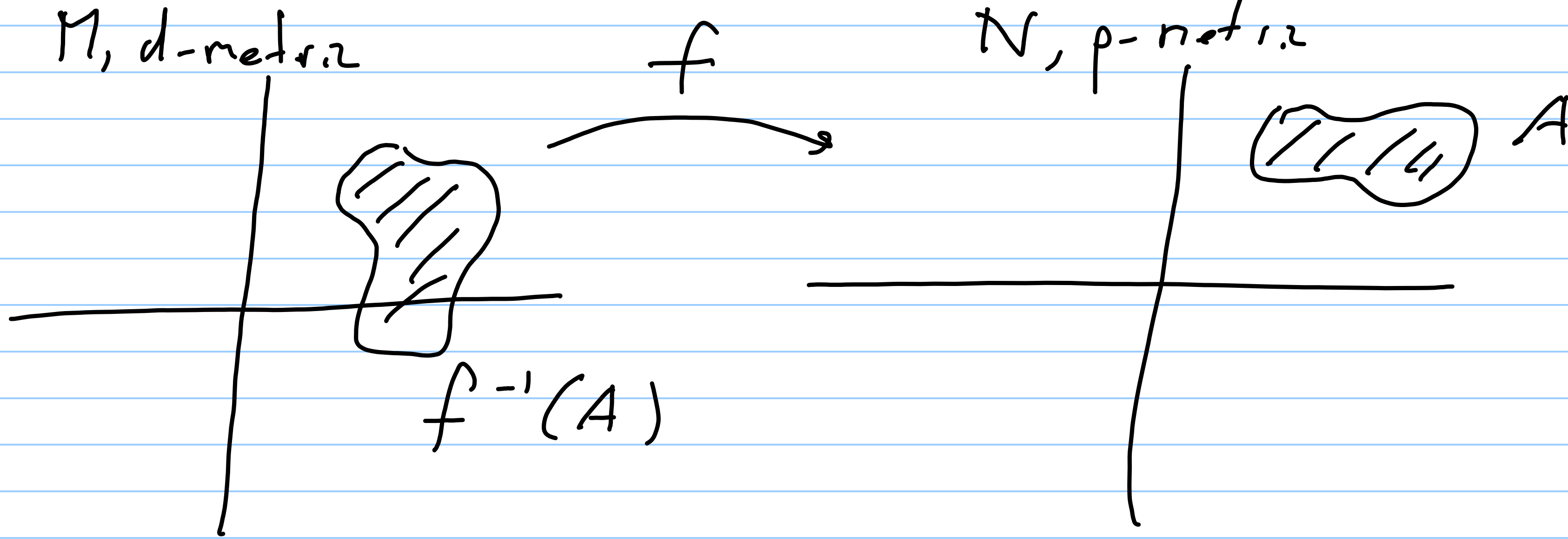
Picture in $\mathbb{R}^2$:



$$f(B_\delta^d(x)) \subset B_\varepsilon^\rho(f(x))$$

We can express this in terms of the inverse image of f, which is defined for $A \subset N$ as follows:

$$f^{-1}(A) := \{ x \in M : f(x) \in A \}.$$



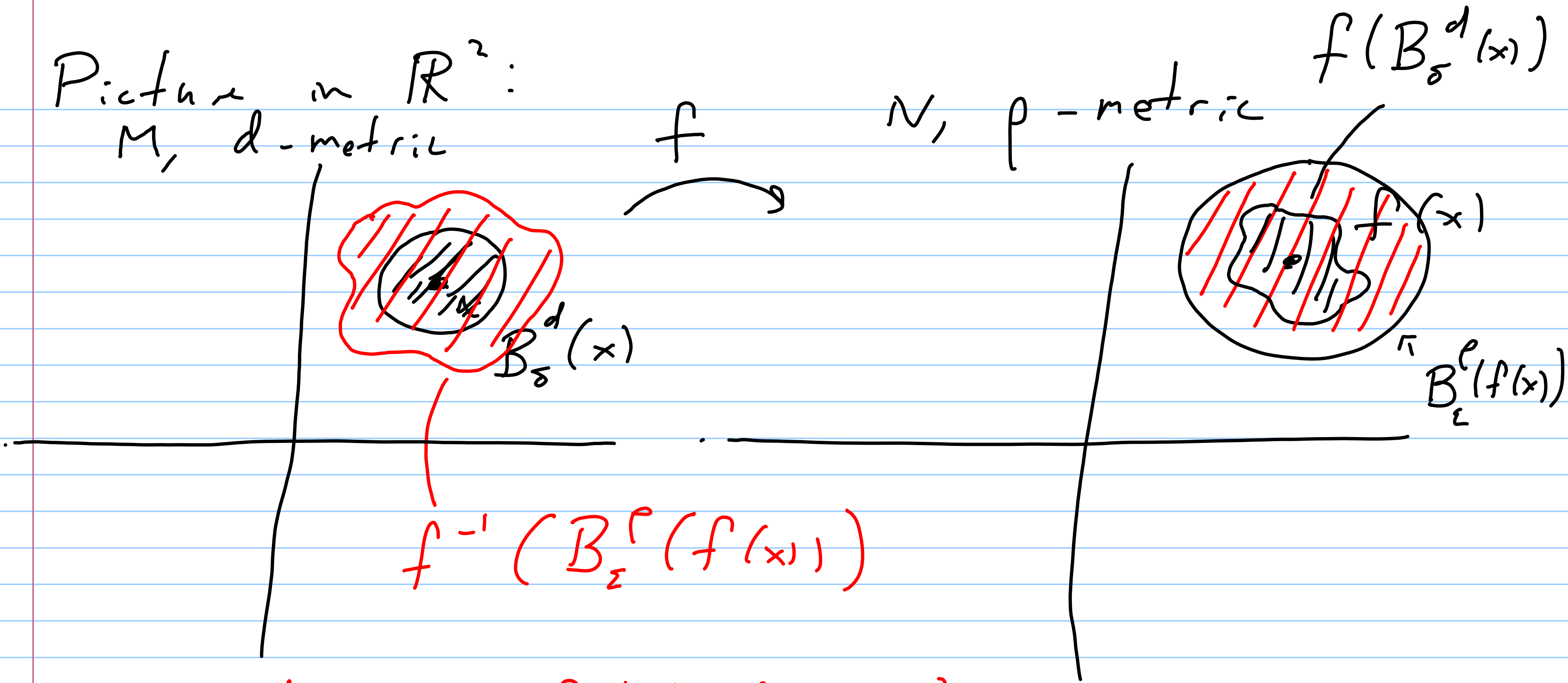
We see that

$$f(B^d_\delta(x)) \subset B^\rho_\varepsilon(f(x))$$

is equivalent to

$$B^d_\delta(x) \subset f^{-1}(B^\rho_\varepsilon(f(x)))$$

Picture in $\mathbb{R}^2$:



$$B_\delta^d(x) \subset f^{-1}(B_\varepsilon^\rho(f(x))).$$