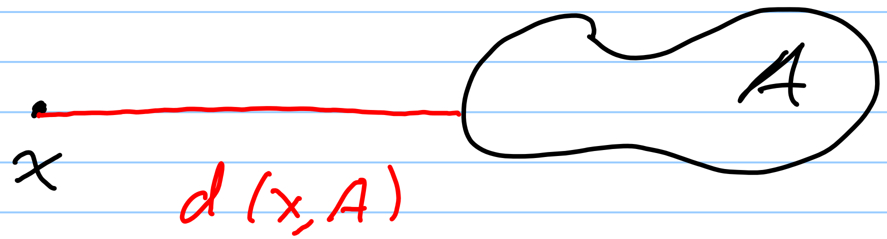
Distance from a Point to a Set

Definition

Given a nonempty set A and a point $x \in M$, we define the distance from x to A by

$$d(x, A) := \inf \{ d(x,a) : a \in A \}.$$



Proposition 5.3

$$d(x, A) = 0 \text{ iff } x \in \bar{A}.$$

Proof

We've seen from the definition of infimum that $d(x,A) = 0$ iff there is a sequence $(a_n) \subset A$ so that $a_n \to x$ in M, but this means $x \in \bar{A}$ (by Corollary 4.11). $\square$

Notice that if we fix a set A, we can define a function on M,

$$f(x) = d(x, A).$$

Intuitively, we expect f to be continuous and this turns out to be the case. In fact:

<u>Proposition 5.4</u>

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

Proof

First, by the triangle inequality

$$d(x,a) \leq d(x,y) + d(y,a)$$

for any $a \in A$. By definition

$$d(x,A) \leq d(x,a) \qquad \forall\, a \in A$$

so $d(x,A) \leq d(x,y) + d(y,a)$. Now we can take the infimum over all $a \in A$ (noting that a only appears on the right-hand side) to see that

$$d(x,A) \leq d(x,y) + d(y,A)$$

and this implies

$$d(x,A) - d(y,A) \leq d(x,y).$$

By symmetry we could have shown

$$d(y,A) - d(x,A) \leq d(x,y),$$

and these two inequalities together give the claim. $\square$

Suppose $f : M \to \mathbb{R}$ is continuous, and consider the set

$$E = \{ x \in M : f(x) = 0 \}.$$

Let's check that E is closed. Suppose $(x_n) \subset E$ converges to $x \in M$. By definition $f(x_n) = 0 \ \forall n \implies$ (by continuity) that

$$f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 0 = 0$$

$\implies x \in E \implies E$ is closed.

Conversely, if E is any closed set in M then we can express it as the zero set of $f(x) = d(x, E)$. I.e.

$$E = \{x \in M : d(x, E) = 0\}.$$

This provides the following characterization: E is closed iff $E = f^{-1}(\{0\})$ for some continuous function $f : M \to \mathbb{R}$. This gives us a direct correspondence between the closed subsets of M and the continuous functions $f: M \to \mathbb{R}$.

Since the open sets are complements of the closed sets, we can likewise view this as a correspondence between the open subsets of M and the continuous functions on M.